Reminders

- Do use DSs: today, 3:00pm, Scott
- I am working on the exam. There are a lot of them! Will publish scores and answers probably on Thursday.
- Tomorrow is election day. We will have class, however.
- Of course you know we switched to Mountain Standard Time

Quick Review

- $\sum_{k=0}^{\infty} a_n$ converges absolutely if $\sum_{k=0}^{\infty} |a_n|$ converges.
- absolute convergence implies convergence!

\[ \text{the converse does does not hold!} \]

- **Absolute Ratio Test** for $\sum_{n=1}^{\infty} a_n$:

- Suppose

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho \]
Then:

\[
\begin{aligned}
  &\text{if } \rho < 1 \text{ then } \sum_{n=1}^{\infty} a_n \text{ converges (absolutely)} \\
  &\text{if } \rho > 1 \text{ then } \sum_{n=1}^{\infty} a_n \text{ diverges} \\
  &\text{if } \rho = 1 \text{ then the test is inconclusive}
\end{aligned}
\]

• Basically: The series behaves like a geometric series with ratio \( \rho \).
9.6 Power Series

- You can think of a power series as a polynomial with infinitely many terms:

\[ s(x) = \sum_{k=0}^{\infty} a_k x^k. \] (1)

- If the upper limit was \( n \) instead of infinity we’d have an ordinary polynomial of degree \( n \). In other words, the partial sums of a power series are ordinary polynomials.

- We can now ask the following Questions:
  1. For what values of \( x \) does the power series converge?
  2. Can we get an expression for the sum of the series in terms of \( x \) if the series does converge?
  3. Can we combine (add, subtract, multiply, divide, compose) power series, or differentiate or integrate them, just like polynomials?

- **Definition:** The convergence set (CS for short) of (1) is the set of all real numbers \( x \) for which the power series converges.
Example:

\[ s(x) = \sum_{k=0}^{\infty} x^k \]

- This is an ordinary geometric series.
- The absolute ratio test turns into

\[ \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \frac{|x^{n+1}|}{|x^n|} = |x| \]

and the series will converge (to \( \frac{1}{1-x} \)) if and only if \( |x| < 1 \).
- Of course, we knew that already!
Example:

\[ s(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \]
Example:

\[ s(x) = \sum_{k=0}^{\infty} k! x^k \]
These examples sum up all possibilities:

- The convergence set of

\[ s(x) = \sum_{k=0}^{\infty} a_k x^k \]

is

(i) the single point \( x = 0 \), or

(ii) an interval of the form \((-R, R)\), with \( R > 0 \), plus possibly one or both end points, or

(iii) the set of all real numbers.

- In the cases (i), (ii), (iii) we say that the radius of convergence of the power series is 0, \( R \), or infinity, respectively.

Every power series has a convergence set of one of those three types. **There are no other possibilities.**

This is not obvious, and, I think, actually counter-intuitive!

- It is worth our while to study this fact closely, to see why, and not just, that it is true.

- So here is the proof from the textbook.

- Suppose

\[ s(x) = \sum_{n=0}^{\infty} a_n x^n \]
converges for
\[ x = x_1 \neq 0. \]

• Then
\[ \lim_{n \to \infty} a_n x_1^n = 0 \]
and there exists a number \( N \) such that
\[ |a_n x_1^n| < 1 \]
for all \( n > N \).

• Suppose that
\[ |x| < |x_1|. \]
Then
\[ |a_n x^n| = |a_n x_1^n| \left| \frac{x}{x_1} \right|^n < \left| \frac{x}{x_1} \right|^n < 1. \]

• Now note that
\[ \sum \left| \frac{x}{x_1} \right|^n \]
is a geometric series with ratio
\[ \left| \frac{x}{x_1} \right| < 1. \]
It converges, and by the ordinary comparison
test so does
\[ \sum_{n=0}^{\infty} |a_n x^n|. \]

• So, if \( s(x) \) converges for \( x = x_1 \), it also con-
   verges for every \( x \) with
   \[ |x| < |x_1|. \]

• By the same token, if it diverges for \( x = x_2 \)
  then it diverges for all \( x \) with
  \[ |x| > |x_2| \]
  since if it did converge for \( x \) it would also have to converge for \( x_2 \).

• This establishes our statement about the three
  kinds of convergence sets.

• Note that we actually established absolute con-
   vergence.

• A power series converges absolutely in
  the interior of its convergence set!

• Let’s do more examples!
\[ s(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}. \]
\[ s(x) = \sum_{n=1}^{\infty} \frac{x^n}{n \times 2^n}. \]
Shift of Origin

- The point $x = 0$ is special for a power series. It is the center of the convergence set.

- Of course, we can do a **shift of origin**, or a **change of variable**:

- The series

  $$s(x) = \sum_{n=1}^{\infty} a_n(x - a)^n$$

  is a power series in $x - a$, centered at $x = a$.

- Example:

  $$s(x) = \sum_{n=2}^{\infty} \frac{(x + 2)^n \ln n}{n \times 3^n}.$$ 

- Find the convergence set.
Example:

\[ s(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k + 1)!} \]

\[ = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \]