Math 1220-3, 7, 90 — Fall 2020

Notes of 10/26/20

Reminder

• DS today, 3:00, Scott

• Exam 2 on Chapters 7 and 8 on Friday and Saturday. Same format as Exam 1. More
detailed announcement is forthcoming.

• We will spend Tuesday and Wednesday of this week reviewing.

Quick Review

• The geometric series

\[ \sum_{n=0}^{\infty} r^n \]

converges if \(|r| < 1\) and diverges if \(r \geq 1\).

• For a geometric series we have \(a_n = r^n\) and hence

\[ \frac{a_{n+1}}{a_n} = r. \]

• Last week we discussed the Ratio Test for positive series, see page 471 textbook.
• Let $\sum_{n=0}^{\infty} a_n$ be a positive series and suppose that

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \rho.$$

Then:

(i) if $\rho < 1$ the series converges
(ii) If $\rho > 1$ (or $\lim \frac{a_{n+1}}{a_n} = \infty$) the series diverges.
(iii) If $\rho = 1$ the test is inconclusive.
9.5 Alternating Series

- We start with a simple example.
- We know that the harmonic series
  
  \[
  1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \ldots
  \]
  
  diverges.
- What about the alternating harmonic series:

  \[
  1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \ldots = \ln 2
  \]
• A series is **alternating** if the signs of its terms alternate. It is of the form

\[ a_1 - a_2 + a_3 - a_4 + a_5 - \ldots \]

where \( a_n > 0 \) for all \( n \).

\[ \text{As usual, finitely many exceptions do not matter for convergence.} \]

**Alternating Series Test**

The alternating series

\[ S = a_1 - a_2 + a_3 - a_4 + a_5 - \ldots \]

where \( a_n > a_{n+1} > 0 \)

converges if and only if

\[ \lim_{n \to \infty} a_n = 0 \]

• Moreover, we have the error bound

\[ |S - S_n| < a_{n+1} \]
Example

\[ S = \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{2^n} \Rightarrow \frac{2^n}{(\ln 2) 2^n} \Rightarrow \frac{2}{(\ln 2)^2} \geq n \]

\[ a_1 = \frac{1^2}{2^1} = \frac{1}{2} \text{ converges} \]

\[ a_2 = \frac{4}{4} = 1 > a_1 \]

\[ a_3 = \frac{9}{8} > a_2 \]

\[ a_4 = \frac{16}{16} < a_3 \]

\[ a_5 = \frac{25}{32} < a_4 \]

\[ \sum \frac{n^2}{2^n} \text{ converges} \]

Ratio Test

\[ \frac{(n+1)^2 2^n}{2^{n+1} n^2} = \frac{(n+1)^2}{n^2 \cdot 2} \Rightarrow \frac{1}{2} \]
Absolute and Conditional Convergence

- We say that the series \( \sum u_n \) converges absolutely if \( \sum |u_n| \) converges.

- We’ll see momentarily that absolute convergence implies convergence.

- The converse does not hold!

- A series that converges but does not converge absolutely is said to converge conditionally.

  alternating harmonic series

  converges

  but does not converge absolutely
\[ S = \sum_{n=1}^{\infty} \frac{\cos n!}{n^2} \]

\[ n! = 1, 2, 6, 24, 120, 720, 5040, \ldots \]

\[ |\cos n!| < 1 \]

\[ \left| \frac{\cos n!}{n^2} \right| < \frac{1}{n^2} \]

\[ \sum \frac{1}{n^2} \text{ converges} \]

\[ \Rightarrow \sum \left| \frac{\cos n!}{n^2} \right| \text{ converges} \]

\[ \Rightarrow \sum \frac{\cos n!}{n^2} \text{ conv} \]
• It’s not obvious that absolute convergence implies convergence, and it’s a little tricky to see why. Here is the argument from the textbook:

• Suppose

\[ \sum_{n=1}^{\infty} |u_n| \]

converges. We want to show that

\[ \sum_{n=1}^{\infty} u_n \]

converges.

• Define

\[ v_n = u_n + |u_n| = \begin{cases} 0 & \text{if } u_n \leq 0 \\ 2u_n & \text{if } u_n > 0 \end{cases} \]

Then

\[ u_n = v_n - |u_n| \]

and

\[ \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} v_n - \sum_{n=1}^{\infty} |u_n| \]

• Both limits on the right exist. \( \sum |u_n| \) by assumption, and the other sum by the comparison test since

\[ 0 \leq v_n \leq 2|u_n|. \]

• Thus the limit on the left also exists.
• Thus we can establish convergence of a series by applying our convergence tests to the non-negative series of absolute values.

• In particular we have the **Absolute Ratio Test**

• Suppose \( S = \sum_{n=1}^{\infty} u_n \) is a series of non-zero terms.

• Suppose

\[
\lim_{n \to \infty} \frac{|u_{n+1}|}{|u_n|} = \rho
\]

(i) if \( \rho < 1 \) the series converges

(ii) If \( \rho > 1 \) (or \( \lim \frac{a_{n+1}}{a_n} = \infty \)) the series diverges (why?).

(iii) If \( \rho = 1 \) the test is **inconclusive**.
Rearrangements

One cool and bizarre feature of conditional convergent series is that they can be rearranged to give different limits!

- This does not hold for finite sums!
- For example:

\[
5 = 4 - 2 + 3 \\
= 4 + 3 - 2 \\
= 3 - 2 + 4 \\
= 3 + 4 - 2 \\
= -2 + 3 + 4 \\
= -2 + 4 + 3
\]

- As an example, consider the alternating harmonic series, and tell you me your favorite number. I’ll re-arrange the series to make its sum your favorite number.

\[
1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \quad = \ln 2 \\
1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2} + \ldots - \frac{1}{4} + \ldots
\]