Reminder

- hw 8 closes Sunday night.
- DS today:
  - 1:00 Liz
  - 2:00 Scott
  - 3:00 Liz
- DS on Sunday, 7:00pm, Liz

Review

- Recall

\[ S = \sum_{n=0}^{\infty} a_n = \lim_{n \to \infty} S_n \]

where

\[ S_n = \sum_{k=0}^{n} a_k \]

- A series is a sum with infinitely many terms.
- We are collecting convergence criteria.
- So far we have, for geometric series,
\[
a \sum_{k=0}^{\infty} r^k \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}
\]

- and the **integral test** for series with positive terms, where \( a_n = f(n) \) and \( f \) is continuous and non-increasing:

\[
\sum_{n=0}^{\infty} a_n \text{ converges } \iff \int_{1}^{\infty} f(x)dx \text{ converges.}
\]

- We also have the **\( p \)-test**, which is a special case of the integral test:

\[
\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges } \iff p > 1.
\]

An often underappreciated major fact is that finitely many exceptions to the series do not matter.

- For example:

\[
15 - \pi + 10^e + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots =
\]
9.4 More tests for positive series

(Ordinary) Comparison Test

• Assume

\[ 0 \leq a_n \leq b_n, \quad n = 0, 1, 2, 3 \ldots \]

Then the following are true:
- if \( \sum_{n=0}^{\infty} b_n \) converges, then so does \( \sum_{n=0}^{\infty} a_n \)
- if \( \sum_{n=0}^{\infty} a_n \) diverges, then so does \( \sum_{n=0}^{\infty} b_n \)

• For example, consider

\[
\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{1}{n^2 + 1},
\]
\[ \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{n}{n^2 + 1}, \]

and

\[ \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n}{2^n(n + 1)}. \]
The Limit Comparison Test

• Assume

\[ a_n \geq 0, \quad b_n > 0, \quad n = 0, 1, 2, \ldots \]

and

\[ \lim_{n \to \infty} \frac{a_n}{b_n} = L \]

Then

– If \( 0 < L < \infty \) then \( \sum_{n=0}^{\infty} a_n \) and \( \sum_{n=0}^{\infty} b_n \) either both converge or both diverge. This is pretty plausible, but there is an \((\epsilon, N)\) proof in the textbook on page 470.

• It is also true that if \( L = 0 \) and \( \sum_{n=0}^{\infty} b_n \) converges then so does \( \sum_{n=0}^{\infty} a_n \).
Example:

\[ S = \sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{3n - 2}{n^3 - 2n^2 + 11} = \]
• Example:

\[ S = \sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2 + 19n}} \]
Here is a trickier example

$$S = \sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{\ln n}{n^2}$$

Try:

$$b_n = \frac{1}{n^2} \quad \text{and} \quad b_n = \frac{1}{n}$$
The Ratio Test

- So the trick is to find a suitable comparison series
- The ratio test compares a sequence with itself.
- Let $\sum_{n=0}^{\infty} a_n$ be a positive series and suppose

$$L = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \rho.$$ 

Then:
- If $\rho < 1$ the series converges.
- If $\rho > 1$ or $\rho = \infty$ the series diverges
- If $\rho = 1$ the test is inconclusive.

- This is eminently plausible: in the limit the series behaves like a geometric series with $r = \rho$.
- There is a more rigorous proof in the textbook.
\[ S = \sum_{n=0}^{\infty} \frac{2^n}{n!} \]

\[ S = \sum_{n=0}^{\infty} \frac{2^n}{n^{20}} \]
• What about $\sum_{n=0}^{\infty} R(n)$ where $R$ is any rational function?