Notes of 10/21/20

Announcements

• DS today, 5:00 pm, Scott
• There is a slight inconsistency between our notes and the textbook. We define

\[ S_n = \sum_{k=0}^{n} a_k \]

whereas the textbook defines

\[ S_n = \sum_{k=0}^{n-1} a_k \]

• This does not change the definitions and results, it’s just a small difference in notation.

Quick Review

• A sequence is of the form

\[ a_0, a_1, a_2, \ldots \]

• Defining the partial sums

\[ S_n = \sum_{k=0}^{n} a_k \]
we say that the series

$$\sum_{n=0}^{\infty} a_n$$

sums (or converges) to

$$S = \sum_{n=0}^{\infty} a_n = \lim_{n \to \infty} S_n$$

if the limit exists.
• In particular we considered geometric series and saw that

\[ a \sum_{k=0}^{n} r^k = \frac{a(1 - r^{n+1})}{1 - r} \]

and

\[ a \sum_{k=0}^{\infty} r^k \]

converges if and only if

\[ |r| < 1 \]

in which case

\[ \lim_{n \to \infty} a \sum_{k=0}^{n} r^k = \frac{a}{1 - r} = a \sum_{k=0}^{\infty} r^k. \]
Collapsing Series

- Also called **Telescoping Series**
- Occur amazingly often. We’ll do just one example:

\[
S = \sum_{n=1}^{\infty} \frac{1}{n(n + 1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{20} + \ldots =
\]
Positive Series

• Our focus today and Friday will be on series with positive terms.

• A series
\[ \sum_{n=0}^{\infty} a_n \]

is

- **positive** if \( a_n > 0 \),
- **non-negative** if \( a_n \geq 0 \),
- **negative** if \( a_n < 0 \),
- **non-positive** if \( a_n \leq 0 \),

for all \( n = 0, 1, 2, \ldots \).

• Clearly, the partial sums of non-negative series converges if and only if its partial sums are bounded above.

Note that finitely many negative terms do not affect convergence and the above statement is still true.
• Example: The harmonic series diverges

• On the other hand, the series

\[ S = \sum_{n=0}^{\infty} \frac{1}{n!} \]

converges
• One of the major organizing principles in Calculus is that integrals behave like sums.

• One example of the correspondence is the **Integral Test** (textbook, page 464):

• Let \( f \) be continuous and non-increasing on the interval \([1, \infty)\), and let

\[
a_n = f(n) \geq 0
\]

Then the series

\[
\sum_{n=1}^{\infty} a_n
\]

converges if and only if the integral

\[
\int_{1}^{\infty} f(x)dx
\]

converges.

• There is a nice picture (Fig. 1, p. 465), and a detailed proof, in the textbook. We’ll just illustrate the essential ideas.

• The key fact is that

\[
\sum_{k=2}^{n} a_k \leq \int_{1}^{n} f(x)dx \leq \sum_{k=1}^{n-1} a_k.
\]
Figure 1. Figure 1, page 465, textbook.
\textbf{$p$-series test}

- An application of the integral test is the $p$-series test:

  The $p$-series

  \[
  \sum_{k=1}^{\infty} \frac{1}{k^p}
  \]

  converges if $p > 1$ and diverges if $p \leq 1$. 

Example 3, p. 466: Does

$$\sum_{k=4}^{\infty} \frac{1}{k^{1.001}}$$

converge or diverge?
Example 3, p. 466: Does

\[ \sum_{k=2}^{\infty} \frac{1}{k \ln n} \]

converge or diverge?
Estimating Limits

- It is a **major idea** that the sum of a series can be approximated by a partial sum.

- Another major idea is that under the same assumptions as before (\(f\) non-increasing, continuous, \(a_n = f(x_n)\)) we can bound the error by an integral.

- Specifically:

\[
\sum_{k=0}^{\infty} a_k \approx S_n = \sum_{k=0}^{n} a_k
\]

and

\[
0 \leq \sum_{k=0}^{\infty} a_k - \sum_{k=0}^{n} a_k \leq \int_{n}^{\infty} f(x) \, dx.
\]
• We may know that a series converges without knowing the actual limit.

• For example, suppose

\[ S = \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}. \]

• We know that the series converges.

• We can estimate \( S \) by a partial sum.

• For example

\[ S \approx \sum_{k=1}^{100} \frac{1}{k^{3/2}} \approx 2.41287\ldots \]

• We can bound the error by an integral:

\[ \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} - \sum_{k=1}^{100} k^{3/2} \leq \int_{100}^{\infty} \frac{1}{x^{3/2}} \, dx = -2x^{-1/2}\bigg|_{100}^{\infty} = 0.2 \]

• It’s beyond our scope to compute the actual sum of the series, but it turns out that

\[ \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} = \zeta\left(\frac{3}{2}\right) = 2.612375\ldots \]

• Here \( \zeta \) denotes the infamous Riemann zeta function.
• Hence the difference between the sum of the series and its partial sum approximation is

\[ \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} - \sum_{k=1}^{100} \frac{1}{k^{3/2}} = 2.612375 - 2.41287 = 0.1995 \]

which is very close to 0.2.

• Of course, an error of 0.2 is still pretty large, this example just illustrates the idea.