Reminder

- DS today, 5:00pm, Scott

8.3 Improper Integrals

- An integral
  \[
  \int_{a}^{b} f(x)\,dx
  \]
  is improper if \( a, b, \) or \( f \) are infinite.

- Example

\[
A = \int_{0}^{\infty} e^{-x}\,dx
\]

\[
A(b) = \int_{0}^{b} e^{-x}\,dx = -e^{-x}\bigg|_{0}^{b} = -e^{-b} + 1
\]

\[
A = \lim_{b \to \infty} A(b) = \lim_{b \to \infty} (1 - e^{-b})
\]

\[
A = 1
\]
Here are the general definitions for integrals with infinite limits of integrations:

\[
\int_a^\infty f(x)dx = \lim_{b \to \infty} \int_a^b f(x)dx
\]

\[
\int_{-\infty}^b f(x)dx = \lim_{a \to -\infty} \int_a^b f(x)dx
\]

\[
\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^0 f(x)dx + \int_0^{\infty} f(x)dx
\]

All limits involved need to exist for the definition to apply.

If the limits exist we say that integral converges, otherwise it diverges.
\[ I = \int_{-\infty}^{0} xe^{-x^2} \, dx. \]

\[
= \lim_{a \to -\infty} \int_{a}^{0} xe^{-x^2} \, dx
\]

\[
= \lim_{a \to -\infty} \int_{a}^{0} \frac{u}{2} e^{-u} \, du
\]

\[
= \frac{1}{2} \lim_{a \to -\infty} \left[ -e^{-u} \right]_{a}^{0}
\]

\[
= \frac{1}{2} \lim_{a \to -\infty} (-1 + e^{-a^2}) = -\frac{1}{2}
\]

\[-(-\infty)^2 = -\infty^2
\]

\[-(-2)^2 = -4\]
\[
I = \int_{0}^{\infty} \sin x \, dx.
\]

\[
I = \lim_{b \to \infty} -\cos x \bigg|_{0}^{b} = \lim_{b \to \infty} (-\cos b + 1)
\]

\[
\lim_{b \to \infty} \cos b \quad \text{DNE}
\]
\[ I = \int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx = \int_{-\infty}^{0} \frac{1}{1+x^2} \, dx + \int_{0}^{\infty} \frac{1}{1+x^2} \, dx \]

**Figure 1.** Graph of \( f(x) = \frac{1}{1+x^2} \) and its antiderivative.

- Figure 1 shows the graphs of \( f(x) = \frac{1}{1+x^2} \) and its antiderivative.

\[
I = \arctan x \bigg|_0^\infty + \arctan x \bigg|_0^{-\infty} \\
= \arctan 0 - \arctan (-\infty) + \arctan (\infty) - \arctan 0 \\
= 0 - \left( -\frac{\pi}{2} \right) + \frac{\pi}{2} - 0 \\
= \pi - \left( -\frac{\pi}{2} \right) - \frac{\pi}{2} \\
= \pi
\]
• Generalization of Example 3 from the text-
book.

• What’s the work required to lift an object
from here to infinity?

• We saw in 1210 that work is the integral of
force with respect to distance.

• Gravity is inversely proportional to the square
of the distance.

• Let \( w \) be the weight (force) on the surface of
the planet, \( R \) the radius of the planet, and \( h \)
the height above the surface of the planet.

• Then the force acting on the object is given
by

\[
F = -\left( \frac{R}{R + h} \right)^2 w.
\]

• Let \( E \) be the work required to lift an object
from the surface of the planet to a height \( H \).

• Take the limit as \( H \) goes to infinity.

• Equate the kinetic energy,

\[
E = -\frac{1}{2} mv^2
\]

with that work and solve for the escape ve-
locity \( v \)
\[ W = \int_0^\infty -\frac{R^2}{(R+h)^2} \, w \, dh \quad \left| \frac{1}{Z} - \frac{1}{Z^2} = \frac{1}{Z} \right. \]

\[ = \frac{R^2}{R+h} \int_0^\infty \, c \, dB \]

\[ = 0 - Rw = -Rw \quad g = \frac{32\,ft}{s^2} \]

\[ \frac{1}{2}mv^2 = Rw = Rmg \]

\[ \frac{1}{2}v^2 = Rg \]

\[ v = \sqrt{2Rg} \]

escape velocity
• On earth we have $R = 3960$ miles and $g = 32$ feet per second squared. This gives an escape velocity of

$$v_0 = \sqrt{2gR}$$

$\approx 6.93$ miles/second

$\approx 11.2$ kilometers/second

$\approx$ Mach 33.

• Here is a table of some other escape velocities, taken from the wikipedia (slightly modified). We assume we are at initially at rest at the given location. (Escaping from the sun starting on the orbiting Earth is easier and more complicated.)

<table>
<thead>
<tr>
<th>Location with respect to</th>
<th>$v_0$(km/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Earth</td>
<td>Earth’s Gravity</td>
</tr>
<tr>
<td>Moon</td>
<td>Moon’s Gravity</td>
</tr>
<tr>
<td>Sun</td>
<td>Sun’s Gravity</td>
</tr>
<tr>
<td>Earth</td>
<td>Sun’s Gravity</td>
</tr>
<tr>
<td>Solar System</td>
<td>Milky Way</td>
</tr>
<tr>
<td>Event Horizon</td>
<td>Black Hole</td>
</tr>
</tbody>
</table>
• Improper integrals occur in probability theory.

• A function \( f \) is a **probability density function** if

\[
f(x) \geq 0, \quad -\infty < x < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} f(x) \, dx = 1.\]

• \( \int_{a}^{b} f(x) \, dx \) is the **likelihood** or **probability** that the event \( x \) is in the interval \([a, b]\)

• The **mean** of \( f \) is

\[
\mu = \int_{-\infty}^{\infty} x f(x) \, dx
\]

and the **variance** of \( f \) is

\[
\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) \, dx.
\]

• \( \sigma \) is called the **standard deviation**.

• For example, let \( \lambda > 0 \) be a parameter and let

\[
f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{else} \end{cases}
\]

• Verify that \( f \) is a probability density function, and compute its mean and variance.
\[
\int_0^\infty 2e^{-2x} \, dx = -e^{-2x}\bigg|_0^\infty = 1
\]

\[
M = \int_0^\infty xe^{-2x} \, dx = -xe^{-2x}\bigg|_0^\infty + \int_0^\infty e^{-2x} \, dx
\]
\[
= \left. -\frac{1}{2}e^{-2x}\right|_0^\infty = 0 - \left(-\frac{1}{2}\right) = \frac{1}{2}
\]

Exercise: \( \theta^2 = \frac{1}{2}\)

\[
\lim_{x \to 0} \frac{\sin x - \tan x}{x^2 \sin x}
\]
\[
= \lim_{x \to 0} \frac{\cos x - \frac{1}{\cos^2 x}}{2 \sin x + x^2 \cos x}
\]
\[
= \lim_{x \to 0} \frac{-\sin x - \frac{2 \sin x}{\cos^3 x}}{2 \sin x + 2x \cos x + 2x \cos x - x^2 \sin x}
\]
\[
= \lim_{x \to 0} \frac{-\sin x - \frac{2 \sin x}{\cos^3 x}}{4x \cos x}
\]
\[
\lim_{x \to \infty} \frac{-\cos x - \frac{2 \cos x + 2 \sin x}{\cos x} 3 \cos^2 x \sin x}{2 \cos x + 4 \cos x - 4 \sin x - \frac{d}{dx} x^2 \sin x}
\]

\[= \frac{-1 - \frac{2}{6}}{6} = -\frac{1}{2}\]

\[
\lim_{x \to 0^+} x \cdot \frac{c}{\ln x} = 0
\]

\[
\lim_{x \to 0^+} \ln x \cdot \frac{c}{\ln x} = \lim_{x \to 0^+} \frac{c}{\ln x} \cdot \ln x
\]

\[= e^c\]

\[
\lim_{x \to 0} \frac{x^2 \sin \frac{1}{x}}{\tan x}
\]

\[
\frac{d}{dx} x^2 \sin \frac{1}{x} = 2x \sin \frac{1}{x} + x^2 \cos \frac{1}{x} \cdot (-\frac{1}{x^2})
\]

\[= 2x \sin \frac{1}{x} - \cos \frac{1}{x}\]
\[
-\lim_{x \to 0} \frac{x^2}{\tan x} \leq \lim_{x \to 0} \frac{x^2 \sin \frac{1}{x}}{\tan x} \leq \lim_{x \to 0} \frac{x^2}{\tan x} \to 0
\]

\[
\lim_{x \to 0} \frac{x^2}{\tan x} = \lim_{x \to 0} \frac{2x}{\cos^2 x} = 0
\]