Notes of 10/13/20

Reminder

• 5:00pm, DS today. I will cover for Scott. Click on my zoom link, not his!

8.2 Other Indeterminate Forms

• Recall the Rule of L’Hôpital. Suppose

\[
\lim_{x \to u} f(x) = \lim_{x \to u} g(x) = 0
\]

and

\[
\lim_{x \to u} \frac{f'(x)}{g'(x)}
\]

exists. Then

\[
\lim_{x \to u} \frac{f(x)}{g(x)} = \lim_{x \to u} \frac{f'(x)}{g'(x)}.
\]

\[\lim_{x \to 0} \frac{1}{x} = \infty\]

Numerator and Denominator must both go to zero.

We are differentiating numerator and denominator separately, we are not applying the quotient rule!
• The Rule of L’Hôpital tells us how to handle the indeterminate expression or indeterminate form \( \frac{0}{0} \)

• The other indeterminate forms are

\[ \frac{\infty}{\infty}, \ 0 \times \infty, \ \infty - \infty, \ 0^0, \ \infty^0, \ 1^\infty. \]

• In all of these cases we have two ingredients that approach the given limits and we ask what happens to the expression in the limit.

• All of these cases go back to the Rule of L’Hôpital, in one way or other.

• We can also consider limits that are infinite, or limits as \( x \) approaches \( +\infty \) or \( -\infty \), or that are one-sided.
• We start with a fanciful example. Suppose you and Jeff Bezos start a new job and stay at it forever. You make $20 per hour and JB make $10 per hour. What is the limit of the ratio of your and JB’s lifetime earnings as time goes to infinity?
• In general, suppose
\[
\lim_{x \to u} |f(x)| = \lim_{x \to u} |g(x)| = \infty
\]
and
\[
\lim_{x \to u} \frac{f'(x)}{g'(x)}
\]
exists. Then
\[
\lim_{x \to u} \frac{f(x)}{g(x)} = \lim_{x \to u} \frac{f'(x)}{g'(x)}.
\]

• We already did one example. Here is another: Suppose \( p \) is any polynomial. What is
\[
\lim_{x \to \infty} \frac{p(x)}{e^x}?
\]
This is Example 2 in the textbook.

\[
\frac{p(x)}{e^x} \quad \rightarrow \quad \frac{p'(x)}{e^x} \quad \rightarrow \quad \frac{p''(x)}{e^x}
\]

• How about
\[
\lim_{x \to \infty} \frac{e^x}{p(x)}?
\]
\[
= \infty
\]
\[ 0 \times \infty \]

- The limit can be obtained by converting the product into a quotient and applying the Rule of L'Hôpital.

- Example 5, textbook.

\[
L = \lim_{x \to \pi^-} \tan x \ln \sin x
\]

\[
= \lim_{x \to \pi^-} \frac{\ln \sin x}{\tan x}
\]

\[
= \lim_{x \to \pi^-} \frac{\frac{\cos x}{\sin x}}{\frac{\cos x}{\sin x}}
\]

\[
= \lim_{x \to \pi^-} \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}
\]

\[
= \lim_{x \to \pi^-} \frac{\cos x \cdot \sin^2 x}{\sin x}
\]

\[
= \lim_{x \to \pi^-} \cos x \sin x
\]

\[
= 0
\]
• Converting a difference to a quotient can be tricky!

• Example 6, textbook:

\[
L = \lim_{{x \to 1^+}} \left( \frac{x}{{x - 1}} - \frac{1}{{\ln x}} \right)
\]

\[
\begin{align*}
&= \lim_{{x \to 1^+}} \frac{x \ln x - (x - 1)}{(x - 1) \ln x} \\
&= \lim_{{x \to 1^+}} \frac{\ln x + x \cdot \frac{1}{x} - 1}{\ln x + (x - 1) \cdot \frac{1}{x}} \\
&= \lim_{{x \to 1^+}} \frac{x \ln x}{x \ln x + x - 1} \\
&= \lim_{{x \to 1^+}} \frac{\ln x + 1}{\ln x + 1 + 1} \\
&= \lim_{{x \to 1^+}} \frac{1}{{1 + 1}} = \frac{1}{2}
\end{align*}
\]
\[ 0^0, \infty^0, 1^\infty \]

- Handle these expressions by computing the limit of the logarithm.

After computing that limit, **don't forget to undo the logarithm**!

- Example:

\[
L = \lim_{x \to 0^+} x^x = 1 \\
M = \lim_{x \to 0^+} x \cdot \ln x = \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} \\
M = \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} \\
= -\lim_{x \to 0^+} \frac{x^2}{x} = -\lim_{x \to 0^+} x = 0
\]

\[
\lim_{x \to 0^+} \ln x^x = 0 = \ln \left( \lim_{x \to 0^+} x^x \right) \\
\ln z = 0 \Rightarrow z = 1
\]
What is the graph of $f(x) = x^3$, anyway?
Figure 1. The graph of $y = x^x$.

- Note that the vertical scale does not start at zero.
• A trick question: What about

\[ L = \lim_{x \to 0^+} (1 + x)^x = 1 \]
• More interesting is

\[ L = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e^{\infty} = 1 \]

• This is the factor by which you multiply your investment each year if the interest rate is 100% and the investment is compounded continuously.

• Any guesses?

\[ M = \lim_{n \to \infty} \ln \left(1 + \frac{1}{n}\right)^{\frac{n}{1}} = \ln L \]

\[ = \lim_{n \to \infty} n \ln \left(1 + \frac{1}{n}\right) \]

\[ = \lim_{n \to \infty} \frac{\ln \left(1 + \frac{1}{n}\right)}{\frac{1}{n}} \]

\[ = \lim_{n \to \infty} \frac{-\ln n^2}{1 + \ln n} \]

\[ = \lim_{n \to \infty} \frac{\frac{1 + \ln n}{n}}{1} \]

\[ = \lim_{n \to \infty} \frac{1}{1 + \ln n} = 1 \]
Example 9, textbook.

\[
\lim_{x \to 0^+} (\sin x)^{1/x} = 0
\]
• Not all limits are suitable for the Rule of L’Hôpital.

• Here are some as yet unfamiliar limits that we will study soon:

\[
\lim_{n \to \infty} \frac{1^k + 2^k + \ldots + n^k}{n^{k+1}} = \frac{1}{k+1}
\]

\[
\int_0^1 \frac{1}{x^2} \, dx = \lim_{a \to 0^+} \int_a^1 \frac{1}{x^2} \, dx = \infty
\]

\[
\int_1^\infty \frac{1}{x^2} \, dx = \lim_{b \to \infty} \int_1^b \frac{1}{x^2} \, dx = 1
\]

\[
\int_{-\infty}^{\infty} e^{-x^2} \, dx = 2 \lim_{b \to \infty} \int_0^b e^{-x^2} \, dx = \sqrt{\pi}
\]

• The last three limits are our next subject.

• We’ll get to the first kind after that.
\[ L = \lim_{x \to 0} (\cos x)^{1/x^2} \]

\[ M = \lim_{x \to 0} \ln (\cos x)^{1/x^2} = \lim_{x \to 0} \frac{\ln \cos x}{x^2} = \frac{0}{0} \]

\[ = \lim_{x \to 0} \frac{-\sin x}{\cos x} \cdot \frac{1}{2x} = -\frac{1}{2} \]

\[ L = e^M = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}} \]