8.1 Indeterminate Forms of type $0/0$

- Recall that we began the study of Calculus by defining the derivative of a function $f$ by
  
  $$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$  

- The expression on the right is a quotient where numerator and denominator both approach zero.

- $0/0$ is undefined, but the limit of such a quotient can be anything.

- A quotient where numerator and denominator simultaneously approach zero is called an indeterminate form of type $0/0$.

- It’s not equal to $0/0$, it’s only a notation for the limit!

- We used such an indeterminate form to define the derivative. That’s what got us going in Calculus!
But now, that we know all about derivatives, we can use derivatives to compute indeterminate forms!

- Here is a quick example. We saw in 1210 that

\[
\lim_{x \to 0} \frac{\sin x}{x} = 1.
\]

- What about the quotient of the derivatives?

\[
\lim_{x \to 0} \frac{\frac{d}{dx} \sin x}{\frac{d}{dx} x} = \lim_{x \to 0} \frac{\cos x}{1} = 1.
\]

- We get the same thing! Is that a coincidence?
- Hardly!
The Rule of L’Hôpital, 1696

• Assume

$$\lim_{x \to u} f(x) = \lim_{x \to u} g(x) = 0$$

and

$$\lim_{x \to u} \frac{f'(x)}{g'(x)}$$

exists. Then

$$\lim_{x \to u} \frac{f(x)}{g(x)} = \lim_{x \to u} \frac{f'(x)}{g'(x)}.$$ 

• The constant $u$ often equals zero, but it can be anything, including plus or minus infinity.

⚠️ Note that we differentiate numerator and denominator separately, we are not applying the quotient rule!

• Guillaume Francois Antoine, Marquis de L’Hôpital. 1661-1704, was a French mathematician. He learned the Rule of L’Hôpital from his teacher, Johann Bernoulli, and published it in 1696 in his textbook “Analyse des Infiniment Petits pour l’Intelligence des Lignes Courbes” which translates as “Infinitesimal Calculus with Applications to Curved Lines”. That book is generally considered the first textbook on Calculus.

• Read the wikipedia on L’Hôpital, it’s interesting!
Examples

\[
\lim_{x \to 0} \frac{1 - \cos x}{x} = \lim_{x \to 0} \frac{\sin x}{1} = \frac{0}{1} = 0
\]

\[
\lim_{x \to 0} \frac{\tan(2x)}{\ln(x+1)} = \lim_{x \to 0} \frac{2}{\cos^2 2x} \cdot \frac{1}{x+1} = \lim_{x \to 0} \frac{2}{\cos^2 2x} = \frac{2}{1} = 2
\]
\[ \lim_{x \to 0} \frac{\sin x - x}{x^3} = \lim_{x \to 0} \frac{\cos x - 1}{3x^2} \]

\[ = \lim_{x \to 0} \frac{-\sin x}{6x} \]

\[ = \lim_{x \to 0} \frac{-\cos x}{6} \]

\[ = -\frac{1}{6} \]
Why is it true?

- The textbook has a rigorous proof based on the “Cauchy Mean Value Theorem”. If you are interested see the book for more info.

- We’ll look at a more casual plausibility argument that nonetheless is pretty compelling.

- Suppose that $f$ and $g$ are differentiable, $f(u) = g(u) = 0$ and $g'(u) \neq 0$.

- Then

$$\lim_{x \to u} \frac{f(x)}{g(x)} = \lim_{x \to u} \frac{f(x) - f(u)}{g(x) - g(u)}$$

$$= \lim_{x \to u} \frac{f(x) - f(u)}{x - u} \cdot \frac{x - u}{g(x) - g(u)}$$

$$= \lim_{x \to u} \frac{f(x) - f(u)}{x - u} \cdot \lim_{x \to u} \frac{g(x) - g(u)}{x - u}$$

$$= \frac{f'(u)}{g'(u)}$$

If you are curious think about why this is only a plausibility argument and the proof in the textbook is more complicated.
• Note again that we are differentiating numerator and denominator separately. We are not applying the quotient rule.

• If we did apply the quotient rule we’d evaluate the derivative of the quotient—not its value—and get something different.

• Example:

\[ L = \lim_{x \to 0} \frac{\sin x}{x} = 1 \]

\[ \frac{d}{dx} \left( \frac{\sin x}{x} \right) = \lim_{x \to 0} \frac{\cos x - \sin x}{2x} = 0 \neq 1 \]
We do have to have that both numerator and denominator go to zero.

- Example:

\[
L = \lim_{{x \to 0}} \frac{{x^2 + x}}{1 - \cos x} = \lim_{{x \to 0}} \frac{{2x + 1}}{\sin x}
\]

\[
= \lim_{{x \to 0}} \frac{2}{\cos x}
= 2 \text{ N.D.}
\]
Also, we may apply the Rule of L’Hôpital repeatedly, and still not get anywhere.

\[ L = \lim_{x \to \infty} \frac{e^{-x}}{x-1} = \lim_{x \to \infty} \frac{-e^{-x}}{x^2} = \lim_{x \to \infty} \frac{e^{-x}}{2x^3} = \ldots \]
• Example: Problem 27, page 427, textbook.

Figure 1 shows a circular sector with angle \( t \). We ask what happens to the ratio of the areas of the yellow and the green regions, as the angle \( t \) approaches zero.

• Any guesses?

• Supposing that the radius \( OA \) of the sector shown in the Figure is 1, the indicated points are given by

\[
A = (1, 0) \\
B = (\cos t, 0) \\
C = (\cos t, \sin t) \\
D = (\cos^2 t, \cos t \sin t) \\
O = (0, 0)
\]
• We want to compute

\[ L = \lim_{t \to 0} \frac{\text{Area of Region } BCD}{\text{Area of Region } ABC} = \lim_{t \to 0} \frac{a(BCD)}{a(ABC)} \]

• Note that

\[
\begin{align*}
a(OBC) &= \frac{1}{2} \cos t \sin t \\
a(OAC) &= \frac{t}{2} \\
a(OBD) &= \frac{t}{2} \cos^2 t
\end{align*}
\]

and hence

\[
L = \lim_{t \to 0} \frac{a(BCD)}{a(ABC)} = \lim_{t \to 0} \frac{a(OBC) - a(OBD)}{a(OAC) - a(OBC)}
\]

\[
= \lim_{t \to 0} \frac{1/2 \sin t \cos t - t/2 \cos^2 t}{t/2 - 1/2 \cos t \sin t}
\]

\[
= \lim_{t \to 0} \frac{\sin t \cos t - t \cos^2 t}{t - \cos t \sin t}
\]

• This is an expression whose limit we can compute by the Rule of L’Hôpital:
\[
L = \lim_{{t \to 0}} \frac{\sin t \cos t - t \cos^2 t}{t - \cos t \sin t}
\]

\[
= \lim_{{t \to 0}} \frac{\cos^2 t - \sin^2 t - \cos^2 t + 2t \sin t \cos t}{1 - \cos^2 t + \sin^2 t}
\]

\[
= \lim_{{t \to 0}} \frac{2t \sin t \cos t - \sin^2 t}{2 \sin^2 t}
\]

\[
= \lim_{{t \to 0}} \frac{2 \cos t \sin t - \sin t}{2 \sin t}
\]

\[
= \lim_{{t \to 0}} \frac{2 \cos^2 t - 2t \sin t - \cos t}{2 \cos t}
\]

\[
= \lim_{{t \to 0}} \frac{\cos t - 2t \sin t}{2 \cos t}
\]

\[
= \frac{1}{2}
\]
• Lest you think that the ratio of these two areas is always 1/2, Figure 2 shows the graph of

$$f(t) = \frac{\sin t \cos t - t \cos^2 t}{t - \cos t \sin t}$$

for the interval $$[-\frac{\pi}{2}, \frac{\pi}{2}]$$.

• Queries: What happens if the angle $$t$$ gets large? What does it mean that evidently $$f(t)$$ may be negative?.
\[
\lim_{x \to \infty} \frac{e^{-x}}{1/x} = 0
\]

\[
\lim_{x \to \infty} \frac{1}{e^{-x} / 1/x} = \lim_{x \to \infty} \frac{e^x}{x}
\]

\[
\frac{\infty}{\infty} \quad \text{L'Hôpital's Rule} \quad \frac{e^x}{1} \to \infty
\]

\[
\lim_{x \to a} \frac{e^x}{p(x)} = \infty
\]

\[
\text{Polyn.}
\]

\[
\lim_{x \to a} \frac{p(x)}{e^x} = 0
\]