Math 1220-3, 7, 90 — Fall 2020

Notes of 10/5/20

Announcements

• I am grading Exam 1. Finishing will take me a few days.
• Apologies for the glitch with problem 3.
• DS today at 3.

More on Integration by Parts

• Recall integration by parts:

\[ \int_{a}^{b} uv^\prime dt = uv \bigg|_{a}^{b} - \int_{a}^{b} u^\prime v dt. \]

• Today let’s start by looking at another example of integration by parts which will lead to the concept of a **Taylor series** which we will approach from a different direction and study in great depth later in the semester.

• For any non-negative integer \( n \) we define \( n! \) (\( n \)-factorial) by

\[
\begin{align*}
n! &= \begin{cases} 
1 & \text{if } n = 0 \\
1 \times 2 \times 3 \times \ldots \times n & \text{if } n > 1
\end{cases}
\end{align*}
\]
The following table gives the first few factorials:

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n!$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>24</td>
<td>120</td>
<td>720</td>
<td>5,040</td>
<td>40,320</td>
</tr>
</tbody>
</table>
• It turns out (under suitable assumptions) that for all $n = 0, 1, 2, \ldots$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + R_n$$

where

$$R_n = \int_0^x \frac{(t-x)^n}{n!} f^{(n+1)}(t)dt.$$ 

• The polynomial part of the right hand side in the equation (1) is part of what is called a Taylor Series. They constitute the second major topic (after exponentials and logarithms) of this semester. More about this in a few weeks, today we’ll look at how to get the equation (1) by integration by parts.
• Example:

\[ f(x) = e^x, \quad f^{(n)}(x) = e^x, \quad f^{(n)}(0) = 1. \]

We get

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \]

This is actually a good way to compute an approximation of \( e \).
7.3 Trigonometric Integrals

- Integration by parts is a major concept and technique. By comparison, section 7.3 is a bag of tricks. We will mostly do a bunch of examples.

- Example 1: \( I = \int \sin x \cos^2 x \, dx \)

- Example 2: \( I = \int \sin^3 x \, dx \)
\[ I = \int \sin^5 x \, dx \]
\[ I = \int_{-\pi}^{\pi} \sin^2 x \, dx \]

\[ I = \int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} \, dx \]
\[ I = \int_{-\pi}^{\pi} \sin(nx) \sin(mx) \, dx \] where \( m \) and \( n \) are integers.
What about $\int_{-\pi}^{\pi} \sin^{13} x \, dx$?
• Important idea: **Reduction Formula**

• Define:

\[ I_n = \int_0^1 t^n e^t \, dt. \]

Then

\[ I_0 = \int_0^1 e^t \, dt = e - 1. \]

Moreover,

\[ I_n = \int_0^1 t^n e^t \, dt = t^n e^t \bigg|_0^1 - n \int_0^1 t^{n-1} e^t \, dt. \]

Thus

\[ I_n = e - n I_{n-1}. \]

• This is an example of a reduction formula. We can use it to compute \( I_1, I_2, I_3 \), one after the other:
the expressions look complicated and unpredictable, but something (perhaps) surprising becomes apparent when they are listed numerically:

\[
\begin{align*}
I_1 &= 1 &= 1.000 \\
I_2 &= e - 2 &= 0.718 \\
I_3 &= -2e + 6 &= 0.563 \\
I_4 &= 9e - 24 &= 0.465 \\
I_5 &= -44e + 120 &= 0.396
\end{align*}
\]

In hindsight, of course the integral is always positive, and goes to zero as \( n \) goes to infinity.