The Gateway Arch in St Louis, MO, forms an "inverted weighted catenary". A catenary
is the shape formed by a rope that supports only its own weight. An inverted catenary arch supports only its own weight, purely by compression and without shear. The height of the arc is 630 feet, as is the width of its base.

• Today we’ll learn what a catenary is, among other things.
Reminders

- DS 3:00 pm today, Scott

6.9 Hyperbolic Functions and their Inverses

- The hyperbolic sine, cosine, and tangent, are defined by

\[
\begin{align*}
\sinh x &= \frac{1}{2} (e^x - e^{-x}) \\
\cosh x &= \frac{1}{2} (e^x + e^{-x}) \\
\tanh x &= \frac{\sinh x}{\cosh x}
\end{align*}
\]

- You might say these are just combinations of exponentials, so they don’t deserve treatment as a separate subject.

- However, they come up a lot in applications, and they have a large number of useful properties. We’ll look at some of them.
$\cosh x = \frac{1}{2}(e^x + e^{-x})$ even

$\sinh x = \frac{1}{2}(e^x - e^{-x})$ odd

Figure 2. Graphs of hyperbolic functions.
Figure 3. Graphs of hyperbolic functions.
Why “Hyperbolic”?  

\[ \cos^2 t + \sin^2 t = 1 \]

\[ x^2 + y^2 = 1 \]

\[
(a+b)^2 = a^2 + 2ab + b^2 \\
(a-b)^2 = a^2 - 2ab + b^2
\]

\[
\cosh^2 t = \left( \frac{1}{2} (e^t + e^{-t}) \right)^2 = \frac{1}{4} \left( e^{2t} + 2e^t e^{-t} + e^{-2t} \right)
\]

\[
\sinh^2 t = \frac{1}{4} \left( e^{2t} - 2e^t e^{-t} + e^{-2t} \right)
\]

\[
\cosh^2 t - \sinh^2 t = \frac{1}{4} \cdot 4 e^t e^{-t} = 1
\]

\[ y^2 - x^2 = 1 \]

\[ y^2 = 1 + x^2 \]

\[ y = \pm \sqrt{1 + x^2} \]
Figure 4. Graph of $y^2 - x^2 = 1$.

**Derivatives**

\[
\frac{d}{dx} \sinh x = \frac{d}{dx} \cdot \frac{1}{2} (e^x - e^{-x}) = \frac{1}{2} (e^x + e^{-x}) = \cosh x
\]

\[
\frac{d}{dx} \cosh x = \frac{d}{dx} \cdot \frac{1}{2} (e^x + e^{-x}) = \frac{1}{2} (e^x - e^{-x}) = \sinh x
\]

\[
\frac{d}{dx} \tanh x = \frac{d}{dx} \frac{\sinh x}{\cosh x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}
\]
Derivatives of Inverses

- cosh is not invertible.
- Let’s do sinh.
- Differentiate implicitly:

\[
\sinh\left(\sinh^{-1}x\right) = x
\]

\[
\cosh(\sinh^{-1}x) \frac{d}{dx} \sinh^{-1}x = 1
\]

\[
\frac{d}{dx} \sinh^{-1}x = \frac{1}{\cosh(\sinh^{-1}x)}
\]

\[
\cosh^2z - \sinh^2z = 1
\]

\[
\cosh^2z = 1 + \sinh^2z
\]

\[
\cosh z = \sqrt{1 + \sinh^2z}
\]

\[
z = \sinh^{-1}x
\]

\[
\frac{d}{dx} \sinh^{-1}x = \frac{1}{\sqrt{1 + \sinh^2\sinh^{-1}x}}
\]

\[
= \frac{1}{\sqrt{1 + x^2}}
\]
• We can compute an explicit expression for $\sinh^{-1}$:

\[
y = \sinh x = \frac{1}{2} (e^x - e^{-x}) \quad x = \frac{2}{u} \quad u = e^x
\]

\[
y = \frac{1}{2} (u - \frac{1}{u})
\]

\[
y u = \frac{1}{2} (u^2 - 1)
\]

\[
\frac{1}{2} u^2 - y u - \frac{1}{2} = 0
\]

\[
e^x = u = y \pm \sqrt{y^2 + 1}
\]

\[
x = \ln \left( y + \sqrt{y^2 + 1} \right)
\]

\[
\sinh^{-1} x = \ln (x + \sqrt{x^2 + 1})
\]
\[ \frac{d}{dx} \ln(x + \sqrt{1 + x^2}) = \]

\[ = \frac{1 + \frac{2x}{2 \sqrt{1 + x^2}}}{x + \sqrt{1 + x^2}} \cdot \frac{\sqrt{1 + x^2}}{\sqrt{1 + x^2}} \]

\[ = \frac{\sqrt{1 + x^2} + x}{(x + \sqrt{1 + x^2}) \cdot \sqrt{1 + x^2}} \cdot \frac{1}{\sqrt{1 + x^2}} \]

\[ = \frac{1}{\sqrt{1 + x^2}} \]
A Catenary

- A hanging rope suspended at \( x = -a \) and \( x = a \) assumes the shape of the **catenary**

\[
y(x) = a \cosh \frac{x}{a}
\]

\[ L > 2a \]

**Figure 5.** Catenary with \( a = 1 \).
Find the length of the catenary

\[ y(x) = a \cosh \frac{x}{a}, \quad -a \leq x \leq a. \]

Expectations?

\[ L = \int_{-a}^{a} \sqrt{1 + f'(x)^2} \, dx \]

\[ f(x) = y(x) = a \cosh \frac{x}{a} \]

\[ f'(x) = \sinh \frac{x}{a} \]

\[ L = \int_{-a}^{a} \sqrt{1 + \sinh^2 \frac{x}{a}} \, dx \]

\[ = \int_{-a}^{a} \cosh \frac{x}{a} \, dx = \left[ a \sinh \frac{x}{a} \right]_{-a}^{a} \]

\[ = a \left( \sinh 1 - \sinh (-1) \right) = 2a \sinh 1 \approx 2.65a \]
\[
\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}.
\]

\[
\sin \sin^{-1} x = x
\]

\[
\cos (\sin^{-1} x) \frac{d}{dx} \sin^{-1} x = 1
\]

\[
\frac{d}{dx} \sin^{-1} (x) = \frac{1}{\cos (\sin^{-1} (x))}
\]

\[
= \frac{1}{\sqrt{1 - \sin^2 (\sin^{-1} x)}}
\]

\[
= \frac{1}{\sqrt{1 - x^2}}
\]

\[
\sin^2 + \cos^2 = 1
\]

\[
\cos = \sqrt{1 - \sin^2 x}
\]

\[
\sin^2 (\sin^{-1} x) = \left( \sin (\sin^{-1} x) \right)^2
\]

\[
= x^2
\]
Falling Object

drag prop. to velocity

\[ v' = -g - \lambda v \quad \lambda > 0 \]

up is positive

\[ s' + \frac{s}{50} = 8 \quad e^{t/50} \]

\[ e^{t/50} \cdot s + \frac{s}{50} e^{t/50} = 8 e^{t/50} \quad \int \]

\[ e^{t/50} \cdot s = 400 e^{t/50} + \frac{1}{\lambda} \]

\[ s = 400 + \frac{1}{\lambda} e^{-t/50} \]

\[ s(0) = 50 = 400 + \frac{1}{\lambda} \]

\[ \lambda = 350 \]