6.6 First Order Linear Differential Equations

- Start with a piece of black magic, and then make sense of it later.

- Suppose we want to solve the differential equation problem.

\[
y' = 2 - 3y, \quad y(0) = 1.
\]

\[
y' + 3y = 2
\]

\[
h = e^{\int 3 \, dt} = e^{3t}
\]

\[
y' e^{3t} + 3y e^{3t} = 2e^{3t}
\]

\[
(\mathcal{L}y e^{3t})' = 2e^{3t}
\]

\[
y e^{3t} = \int 2e^{3t} \, dt = \frac{2}{3} e^{3t} + C
\]

\[
y = \frac{2}{3} + C e^{-3t}
\]

**General Solution**

\[
y = \frac{2}{3} + C e^{-3t}
\]
\[ \int 2e^{3t} \, dt \]
\[ = 2 \int e^{3t} \, dt \]
\[ = \frac{2e^{3t}}{3} \]

\[ \frac{d}{dt} e^{3t} = 3e^{3t} \]

\[ y = \frac{2}{3} + Ce^{-3t} \]

\[ y(0) = 1 \]

\[ 1 = \frac{2}{3} + Ce^{-3(0)} \]
\[ = \frac{2}{3} \]

\[ \frac{1}{3} = C \]

\[ y = \frac{2}{3} + \frac{1}{3} e^{-3t} \]

particular solution
A first order linear differential equation is of the form
\[
\frac{dy}{dx} + P(x)y = Q(x).
\]

Here \(y = y(x)\), and \(P\) and \(Q\) are given functions of \(x\) (and not \(y\) or \(y'\)).

We want to compute all solutions \(y\), or perhaps just one determined by an initial condition
\[
y(a) = y_0.
\]

The independent variable is often \(t\) (for time) instead of \(x\).

Let’s start with a fanciful example chosen to give particularly simple numbers. (This is like Problem 1 on page 358).

You jump out of an airplane. Suppose \(v(t)\) is your downward speed. You use a parachute that causes a drag which is proportional to your speed. Suppose the constant of proportionality is 1, and gravity causes a downward acceleration of 32 feet per second squared. Your initial velocity is zero. What is your velocity as a function of time \(t\)? (In this problem, the positive direction is down.)

We get the initial value problem:
\[
v' = 32 - v, \quad v(0) = 0.
\]
- How do we translate this to our **first order** linear differential equation of the form:

\[
\frac{dy}{dt} + P(t)y = Q(t).
\]

**Integrating Factor**

\[
h = e^{\int P(t) dt}
\]

\[
h' = P(t) e^{\int P(t) dt}
\]

\[
h = \int P(t) e^{\int P(t) dt} dt
\]

\[
y' + Py = Q
\]

\[
y'h + Pyh = Qh
\]

\[
y'h + h'y = Qh
\]

\[
(yh)' = Qh
\]

\[
yh = \int Qh dt
\]

\[
y = \frac{\int Qh dt}{h}
\]

\[
y = \frac{\int Q(t) e^{\int P(t) dt} dt}{e^{\int P(t) dt}}
\]
• Return to

\[ y' + P(x)y = Q(x). \]

• We want to integrate this differential equation with a factor that makes the right hand side the derivative of a product by the product rule.

\[ v' = 32 - v \quad \quad \quad v(0) = 0 \]

\[ v' + v = 32 \quad +v \]

\[ h = e^{\int 1 \, dt} = e^t \]

\[ (ve^t)' = 32e^t \quad \quad h \]

\[ \int 32e^t \, dt \]

\[ ve^t = 32e^t + C \]

\[ v = 32 + Ce^{-t} \quad \text{general sol} \]

\[ 0 = 32 + Ce^{-0} \]

\[ -32 = C \]
\[ v = 32 + Ce^{-t} \]

\[ C = -32 \]

\[ v = 32 - 32e^{-t} \]  \hspace{1cm} \text{particular solution}

\[ t = 0 \quad v = 0 \]

\[ v = 32 - 32e^0 \]

\[ \text{as time } t \uparrow e^{-t} \downarrow \]

\[ \text{as } t \to \infty, \quad v \to 32 \]
• We get the monster formula (p. 355, textbook)

\[ y(x) = e^{-\int P(x)\,dx} \int Q(x) e^{\int P(x)\,dx} \,dx. \]

• Don’t memorize this formula! Instead, follow the procedure we used to derive it. Multiply with \( e^{z(x)} \) where \( z' = P(x) \) and go from there.

• Example 1, textbook.

\[ \frac{dy}{dx} + \frac{2}{x}y = \frac{\sin 3x}{x^2}. \]

\[ h = e^{\int \frac{2}{x} \,dx} \]
\[ = e^{2\ln x} \]
\[ = x^2 \]

\[ (\frac{y}{x^2})' = \frac{y'}{x^2} + \frac{y (x')'}{x^2} \]
\[ = \frac{y'}{x^2} + 2\frac{y}{x} \]

\[ (\frac{y}{x^2})' = \frac{\ln 3x}{x^2} \]
\[ \frac{y}{x^2} = \int \frac{\ln 3x}{x^2} \,dx \]
\[ \frac{y}{x^2} = -\frac{1}{3} \cos 3x + C \]
\[ y = -\cos 3x \cdot \frac{c}{x^2} \]
Electric Circuit

- Consider an electric circuit that has a battery ($E$ volts), a switch, a resistor ($R$ Ohms), and an inductor, (a coil of wire, $L$ Henry) in series.

\[
LI' + RI = E
\]

- Let $I(t)$ be the current in this circuit, assuming the switch is closed at time 0. Compute $I(t)$.

- The relevant initial value problem is

\[
L \frac{dI}{dt} + RI = E, \quad I(0) = 0.
\]
\[
L \frac{dI}{dt} + RI = E \quad \div L
\]
\[
\frac{dI}{dt} + \frac{R}{L} I = \frac{E}{L}
\]
\[
(I \cdot e^{\frac{R}{L} t})' = \frac{E}{L} e^{\frac{R}{L} t} \quad \cdot e^{\frac{R}{L} t}
\]
\[
I e^{\frac{R}{L} t} = \int \frac{E}{L} e^{\frac{R}{L} t} \, dt
\]
\[
I e^{\frac{R}{L} t} = \frac{E}{L} e^{\frac{R}{L} t} \frac{L}{R} + C
\]
\[
I e^{\frac{R}{L} t} = \frac{E}{R} e^{\frac{R}{L} t} + C
\]
\[
I = \frac{E}{R} + C e^{-\frac{R}{L} t}
\]
\[
I(0) = 0
\]
\[
0 = \frac{E}{R} + C e^{0} - \frac{E}{R}
\]
\[
-\frac{E}{R} = C
\]
\[
I = \frac{E}{R} - \frac{E}{R} e^{-\frac{R}{L} t}
\]
• If time permits, Example 2, textbook:

\[
\frac{dy}{dx} - 3y = xe^{3x}, \quad y(0) = 4.
\]

\[
h = \int e^{-3x} dx
\]

\[
= e^{-3x}
\]

\[
(y e^{-3x})' = xe^{3x} \cdot e^{-3x}
\]

\[
(y e^{-3x})' = x
\]

\[
y e^{-3x} = \int x \, dx
\]

\[
y e^{-3x} = \frac{1}{2} x^2 + C
\]

\[
y = \frac{1}{2} x^2 e^{3x} + Ce^{3x}
\]

\[
y = \frac{1}{2} x^2 e^{3x} + 4e^{3x}
\]