Math 1220-3

Notes of 1/17/18

6.3 The natural exponential function

- Recall: we defined the natural logarithm $\ln x$ as
  $$\ln x = \int_1^x \frac{1}{t} dt$$

- This implies immediately, by the FToC, that
  $$\frac{d}{dx} \ln x = \frac{1}{x}.$$  

- $\ln x$ is invertible.

- How do we know that?

- We denoted the inverse of $\ln x$ by $\exp x$ and call that inverse the natural exponential.

- We are used to thinking of the natural exponential as $e^x$ where $e$ is the number whose natural logarithm is 1, i.e.,
  $$\exp(\ln x) = x$$

  $$\ln e = \int_1^e \frac{1}{t} dt = 1,$$  \hspace{1cm} (1)

  or

  $$e = \exp 1.$$
Figure 1. Definition of $e$.

- According to http://www.numberworld.org/digits/E/, as of August 20, 2016, $e$ has been computed to 5 billion digits.

- Here are the first few:

$$e = 2.71828 18284 59045 23536 02874 71352$$

$$66249 77572 47093 69995 95749 66967 62772 40766$$

$$30353 54759 45713 82178 52516 64274 27466 39193$$

$$20030 59921 81741 35966 29043 57290 03342 95260$$

$$59563 07381 32328 62794 34907 63233 82988 \ldots$$
• We need to show that

\[ \exp x = e^x \quad (2) \]

• This follows from our earlier observation that

\[ \ln a^r = r \ln a \]

because

\[ \ln e^x = x \ln e = x = \ln \exp x \quad \Rightarrow \exp x = e^x \]

and (2) follows.

• You may remember from College Algebra that \( e \) can be defined by compounding 100 percent interest continuously. This leads to the definition

\[ e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \]

• We will see later in the semester that the so defined limit gives the same value of \( e \) as we obtained via the area definition (1) of \( e \).
Differentiation of Exponentials and Logarithms

- We can now work with our new differentiation rules:

\[
\frac{d}{dx} \ln x = \frac{1}{x} \\
\frac{d}{dx} e^x = e^x
\]

- Let’s do examples:

\[
\frac{d}{dx} 10^x = \frac{d}{dx} \left(e^{\ln 10}\right)^x = \frac{d}{dx} e^{x \ln 10} = e^{x \ln 10} \cdot \ln 10 = 10^x \ln 10
\]

\[
\frac{d}{dx} b^x = b^x \ln b \quad b > 0
\]

\[
10 = e^{\ln 10}
\]
• Recall the conversion formula for logarithms:

\[
\log_b x = \frac{\ln x}{\ln b}
\]

• why?

\[
\begin{align*}
\log_b x &= x \left(\frac{1}{\ln b}\right) \\
\log_b x &= \log_b (\ln b) \\
\log_b x &= \frac{\ln x}{\ln b}
\end{align*}
\]
\[
\frac{d}{dx} \ln(\sin x) = \frac{\cos x}{\sin x}
\]

\[
\frac{d}{dx} \sin(\ln x) = \frac{\cos(\ln x)}{x} = \frac{1}{x} \cdot \cos(\ln x)
\]

\[
\frac{d}{dx} e^{\sin x} = e^{\sin x} \cdot \cos x
\]

\[
\frac{d}{dx} \sin e^x = (\cos e^x) e^x
\]

\[
\frac{d}{dx} \sin e^{x^2} = (\cos e^{x^2}) e^{x^2} - 2x = 2xe^{x^2} \cos e^{x^2}
\]

\[
\frac{d}{dx} e^{e^x} = e^{e^x} \cdot e^x = e^{x+e^x}
\]
What about integration?

\[ \int e^{-4x} \, dx = -\frac{1}{4} e^{-4x} + C \]

\[ u = -4x \]
\[ du = -4 \, dx \quad \Rightarrow \quad dx = \frac{du}{-4} \]

\[ \int e^{-4x} \, dx = \int e^u \, \frac{du}{-4} = -\frac{1}{4} \int e^u \, du = -\frac{1}{4} e^u + C = -\frac{1}{4} e^{-4x} + C \]

\[ I = \int_0^1 e^{-4x} \, dx = \left[ -\frac{1}{4} e^{-4x} \right]_0^1 = -\frac{1}{4} (e^{-4} - 1) \]

\[ I = \int x^2 e^{-x^3} \, dx = \]

\[ u = -x^3 \]
\[ du = -3x^2 \, dx \]
\[ x^2 \, dx = -\frac{1}{3} \, du \]

\[ I = \int \frac{1}{3} e^u \, du = -\frac{1}{3} e^u + C = -\frac{1}{3} e^{-x^3} + C \]

\[ \int xe^{-x^3} \, dx = \text{hopeless} \]

\[ \frac{e^{-x^3}}{x^2} + C \]

Math 1220-3 Notes of 1/17/18 page 7
Let’s look at the graph of

$$f(x) = e^{-x^2}$$

and compute some derivatives.

Figure 2. $f(x) = e^{-x^2}$ and some derivatives.
\[ f'(x) = -2xe^{-x^2} \]
\[ f''(x) = -2e^{-x^2} + (-2x)(-2x)e^{-x^2} \]
\[ = -2e^{-x^2} + 4xe^{-x^2} \]