6.2 Inverse Functions and their Derivatives

• Recall, we defined (in section 6.1)

\[ \ln x = \int_1^x \frac{1}{t} \, dt. \]  \hfill (1)

• We obtained, as an immediate consequence by the Fundamental Theorem of Calculus:

\[ \frac{d}{dx} \ln x = \frac{1}{x}. \]

• We also established, with our definition, that

\[ \frac{d}{dx} \ln |x| = \frac{1}{x} \]

\[ \ln 1 = 0 \]

\[ \ln ab = \ln a + \ln b \]  \hfill (2)

\[ \ln \frac{a}{b} = \ln a - \ln b \]

\[ \ln x^r = r \ln x \]

• You might say that this is nothing new, you are already familiar with the natural logarithm. However, what is in fact new is our
definition (1). The properties in (2) help establish that our new definition is consistent with our old definition.

- What is that old definition (from College Algebra)? The natural exponential is that exponential that is obtained by compounding 100 percent interest continuously and the natural logarithm is the inverse of that exponential.

- That’s highly unsatisfactory! What does mathematics have to do with paying interest on an investment?

Aside: In WeBWorK, \( \ln x \) and \( \log x \) mean the same thing, i.e., the natural logarithm.

- You can convert from the natural logarithm to any other base \( b (b > 0, b \neq 1) \), including \( b = 10 \), by the formula

\[
\log_b x = \frac{\ln x}{\ln b}.
\]

Aside: Before losing sight of the forest for all the trees, today’s main new insight will be that

\[
\frac{d}{dx} e^x = e^x.
\]

The exponential equals its own derivative!
• This looks very plausible if you consider the graph of the exponential shown in Figure 1.

• The function and its derivative are both positive. As the function gets larger the derivative gets larger too.

• How do we make this more precise?

• The starting point is that the exponential is the inverse of the logarithm:

\[ \ln e^x = x. \]

Differentiating on both sides and using the fact that the derivative of \( \ln x \) is \( 1/x \) gives, by the chain rule,

\[ \frac{d}{dx} \ln e^x = \frac{1}{e^x} \frac{d}{dx} e^x = 1 \quad \implies \quad \frac{d}{dx} e^x = e^x. \]

• It’s that simple, but let’s explore inverse functions in more depth!
Inverse Functions

- What’s an inverse function?

- The inverse of a function \( f \), if it exists, is denoted by \( f^{-1} \) (pronounced \( f \)-inverse) and is defined by

\[
\begin{align*}
    f^{-1}(f(x)) &= x & \text{and} & & f(f^{-1}(y)) &= y
\end{align*}
\]

for all \( x \) in the domain of \( f \) and all \( y \) in the domain of \( f^{-1} \).

This notation is used everywhere on earth. Nonetheless, it is truly awful! Note that in general

\[
    f^{-1}(x) \neq \frac{1}{f(x)}.
\]

- The superscript \(-1\) may denote the inverse or the reciprocal. Which it is depends on the context!

- Life is like that!
• Example

\[ f(x) = 2x + 6 \]
• Example 3:

\[ f(x) = \frac{x}{1 - x}. \]
• We find the inverse of a function $f$ by solving the equation $y = f(x)$ for $x$ and then interchanging $x$ and $y$.

\[ \text{A function may not have an inverse!} \]

• Examples:
- Clearly, the graphs of a function and its inverse must be related.

Solving \( y = f(x) \) for \( x \) does not change the graph.

However, switching \( x \) and \( y \) reflects the graph in the line \( y = x \).

![Graph of a line reflecting in the line \( y = x \).]

**Figure 2.** Reflecting in \( y = x \).
Figure 3. Graphs of $f(x) = 2x + 6$ and its inverse.
Figure 4. Graphs of $f(x) = \frac{x}{1-x}$ and its inverse.

- The graphs for our first two examples are shown in Figures 3 and 4.
- When does a function have an inverse?
- Function: Vertical Line Test
- Inverse: Apply VLT to inverse, or apply horizontal line test to function.
- In particular a function has an inverse if it is
monotonic.

- This means it’s either everywhere increasing, or everywhere decreasing.

- This is true in particular if $f'$ is everywhere positive (increasing), or $f'$ is everywhere negative (decreasing).

In particular, the natural logarithm is monotonic and has an inverse!

$$\frac{d}{dx} \ln x = \frac{1}{x} > 0 \quad (x > 0).$$
Figure 5. Natural Logarithm and Exponential.

- Figure 5 shows the natural logarithm and its inverse.
Derivatives of Inverse Functions

• Start with

\[ f^{-1}(f(x)) = x. \]

• Differentiate, using the chain rule:

\[ (f^{-1})'(f(x))f'(x) = 1. \]

• set \( y = f(x) \) and solve for \( (f^{-1})(y) \):

\[ (f^{-1})'(y) = (f^{-1})'(f(x)) = \frac{1}{f'(x)} \]

• The derivative of the inverse is the reciprocal of the derivative.

• This can be written very suggestively as

\[ \frac{dx}{dy} = \frac{1}{dy/dx}. \]
Example 4.

\[ f(x) = x^5 + 2x + 1, \quad f(1) = 4, \quad (f^{-1})'(4) = ? \]
The derivative of the exponential, revisited

- Denote the inverse of the natural logarithm by \( \exp(x) \). (Strictly speaking, we don’t know yet that \( \exp(x) = e^x \).)

- Start with
  \[
  \ln \exp(x) = x.
  \]

- Differentiate:
  \[
  \frac{d}{dx} \frac{\exp(x)}{\exp(x)} = 1.
  \]

- This gives, again, our desired result:
  \[
  \frac{d}{dx} \exp(x) = \exp(x).
  \]

- The exponential equals its own derivative.

- This makes geometric sense.
Examples

\[ \frac{d}{dx} e^{x^2} = \]

\[ \int 2x e^{x^2} \, dx = \]

• Do again \( \frac{d}{dx} x^x = \)