Announcements

- You should have received an email from me with login info for WeBWorK. The first homework is open, and serves as a review of Math 1210.

- hw's open on Mondays and close 12 days later on Fridays. However, you should finish each homework before the next one opens.

- Thus you should finish hw 1 by Sunday. However, it will be open until a week from today.

- hw 2 will open Monday.

- The key to succeeding in mathematics is to appreciate that it builds on itself. Stay on top of things and don’t fall behind!
• We’ll start with a quick review:

**Chain Rule**

• Recall function composition

\[ f \circ g(x) = f(g(x)). \]

• The Chain Rule can be written as

\[
\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).
\]

• In other words, the **derivative of the composition** is the product of the derivatives.

**Same Derivatives**

• Recall that two functions with the same derivative are the same except they may differ by a constant.

**Fundamental Theorem of Calculus**

\[
\frac{d}{dx} \int_a^x f(t)\,dt = f(x).
\]
6.1 The natural logarithmic function

- We are now starting with the subject of Math 1220 proper.
- I assume you are familiar with exponentials and logarithms, particularly the natural logarithm.

\[ \ln, \quad e^x, \quad e = 2.71828\ldots \]

- The major fact that we will begin to understand today is

\[ \frac{d}{dx} \ln x = \frac{1}{x} \]

or, equivalently,

\[ \int_1^x \frac{1}{t} \, dt = \ln x. \]

- This makes geometric sense, as illustrated in Figure 1.
- Note that our power rule

\[ \int t^r \, dt = \frac{t^{r+1}}{r+1} \quad (+ C) \]

fails if \( r = -1 \). We will see today that

\[ \int_{\frac{1}{x}}^1 t^{-1} \, dt = \int_1^x \frac{1}{t} \, dt = \ln x. \]
Figure 1. ln and its derivative.

- There are many different ways to introduce the derivatives of exponentials and logarithms. In the spirit of Math 1210 we could define

\[
\frac{d}{dx} \ln x = \lim_{h \to 0} \frac{\ln(x + h) - \ln x}{h}.
\]

- It would be hard to compute that limit with what we have learned in Math 1210.

- Instead the textbook proceeds as follows: It
defines

\[ \ln x = \int_{1}^{x} \frac{1}{t} dt. \]

- With that definition it follows immediately from the FToC that

\[
\frac{d}{dx} \ln x = \frac{1}{x}.
\]

- Of course, this definition is useful only if it is consistent with our already existing understanding of the natural logarithm!

- It is, and we will eventually recognize that this is so by showing that the function we defined has all the properties if the natural logarithm with which we are familiar.
• Verifying a bunch of properties of the logarithm will give us the opportunity to practice the differentiation of logarithmic function.

• First example:

\[ \frac{d}{dx} \ln |x| = \frac{1}{x}. \]

\( x > 0 \) \[ \frac{d}{dx} \ln |x| = \frac{d}{dx} \ln x = \frac{1}{x} \]

\( x < 0 \) \[ \frac{d}{dx} \ln |x| = \frac{d}{dx} \ln (-x) = \frac{-1}{-x} = \frac{1}{x} < 0 \]
Next verify that the following properties of the natural logarithm are satisfied.

\[ \ln 1 = 0 \] (i)
\[ \ln ab = \ln a + \ln b \] (ii)
\[ \ln \frac{a}{b} = \ln a - \ln b \] (iii)
\[ \ln a^r = r \ln a \] (iv)

(assuming that \( a, b > 0 \)).

\[ (i) \quad \ln 1 = 0 \]
\[ \ln 1 = \int_{\frac{1}{e}}^{1} \frac{1}{t} \, dt = 0 \]
(ii) \( \ln(ab) = \ln(a) + \ln(b) \).

\[ x = b \]

\[ \ln(ax) = \ln(a) + \ln(x) = \ln(a) + \ln(x) \]

\[ \frac{d}{dx} \ln(ax) = \frac{a}{ax} = \frac{1}{x} = \frac{d}{dx} \ln(x) \]

\[ \Rightarrow \ln(ax) = (\ln(x)) + c \]

\[ x = 1 \quad \ln(a \cdot 1) = 0 + c \]

\[ c = \ln(a) \]
\[ (iii) \quad \ln \frac{a}{b} = \ln a - \ln b. \]

\[ b = x \]

\[ \ln \frac{a}{x} = \ln (a \cdot \frac{1}{x}) = \ln a + \ln \frac{1}{x} = \ln a - \ln x \]

\[ \frac{d}{dx} \ln \frac{1}{x} = \frac{-1/x^2}{1/x} = - \frac{x}{x^2} = - \frac{1}{x} \]

\[ \frac{d}{dx} \frac{1}{x} = \frac{d}{dx} x^{-1} = (-1)x^{-2} = -x^{-2} = - \frac{1}{x^2} \]

\[ \frac{d}{dx} (-\ln x) = - \frac{1}{x} = \frac{d}{dx} \ln \frac{1}{x} \]

\[ \ln \frac{1}{x} = - \ln x + \xi \]

\[ x = 1 \quad \ln 1 = -\ln 1 + \xi \]

\[ \xi = -0 + \xi \]

\[ C = 0 \]

\[ \ln \frac{1}{x} = -\ln x \]
(iv) \( \ln a^r = r \ln a. \)

\[
\frac{d}{dx} e^{nx^r} = \frac{TX^{r-1}}{x^r} = r \frac{1}{x} = x = q
\]

\[
\ln x^r = r \ln x + c \quad x = 1
\]

\[
e^{ln 1^r} = r \ln 1 + c
\]

\[
0 = r \cdot 0 + c \quad c = 0
\]
Examples

Compute \( \frac{d}{dx} \ln x^2 = ? \)
in two different ways.

\[
\frac{d}{dx} \ln x^2 = \frac{2x}{x^2} = \frac{2}{x}
\]

\[
\frac{d}{dx} \ln x^2 = \frac{d}{dx} 2 \ln x = 2 \frac{d}{dx} \ln x
\]

\[= 2 \cdot \frac{1}{x} = \frac{2}{x}\]
\[ \int \frac{2x}{1 + x^2} \, dx = ? = \int \frac{du}{u} \]

\[ U = 1 + x^2 \]

\[ \frac{du}{dx} = 2x \]

\[ du = 2x \, dx \]

\[ C \neq D \]

\[ \frac{d}{dx} \ln(x^2 + 1) = \frac{2x}{x^2 + 1} \quad \checkmark \]

OK
\[ \frac{d}{dx} x^x = ? \]

\[ y = x^x \quad \Rightarrow \quad y' = ? \]

\[ \ln y = x \ln x \]

\[ \frac{y'}{y} = \ln x + x \cdot \frac{1}{x} = (\ln x) + 1 \]

\[ y' = y \left( 1 + \ln x \right) \]

\[ = x^x \left( 1 + \ln x \right) \]
\[ \int \tan x \, dx = \] 

\[ \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx \] 

\[ = - \ln |\cos x| \] 

CBD  \[ \frac{d}{dx} \left( - \ln |\cos x| \right) = - \frac{-\sin x}{\cos x} \] 

\[ = \frac{\sin x}{\cos x} = \tan x \]
\[
\int \frac{x^2 + 1}{x + 1} \, dx = ?
\]