8.3 Improper Integrals

• An integral
  \[ \int_{a}^{b} f(x)\,dx \]

  is improper if \( a, b, \) or \( f \) are infinite.

• Example

\[ y = e^{-x} \]

**Figure 1.** Graph of \( y = e^{-x} \).
\[ A = \int_{0}^{\infty} e^{-x} \, dx \]
Here are the general definitions for integrals with infinite limits of integrations:

\[ \int_a^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx \]

\[ \int_{-\infty}^b f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx \]

\[ \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^{\infty} f(x)\,dx \]

All limits involved need to exist for the definition to apply.

If the limits exist we say that integral \textbf{converges}, otherwise it \textbf{diverges}. 
Figure 2. Graph of $y = xe^{-x^2}$.

$I = \int_{-\infty}^{0} xe^{-x^2} \, dx.$
Figure 3. The sin function.

\[ I = \int_{0}^{\infty} \sin x \, dx. \]
Figure 4. Graphs of $y = \frac{1}{1+x^2}$ and $y = \arctan x$.

$$\int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx =$$
• Generalization of Example 3 from the textbook.

• What’s the work required to lift an object from here to infinity?

• We saw in 1210 that work is the integral of force with respect to distance.

• Gravity is inversely proportional to the square of the distance.

• Let \( w \) be the weight (force) on the surface of the planet, \( R \) the radius of the planet, and \( h \) the height above the surface of the planet.

• Then the force acting on the object is given by

\[
F = -\left(\frac{R}{R+h}\right)^2 mg.
\]

• \( g \) is gravity on the surface of the planet, \( m \) is the mass of the object, and \( w = mg \) is the weight of the object at the surface of the planet.

• Let \( E \) be the work required to lift an object from the surface of the planet to a height \( H \).

• Take the limit as \( H \) goes to infinity.

• Equate the kinetic energy,

\[
E = -\frac{1}{2}mv^2
\]
with that work and solve for the escape velocity \( v \)
- On earth we have $R = 3960$ miles and $g = 32$ feet per second squared. This gives an escape velocity of

$$v_0 = \sqrt{2gR}$$

$$\approx 6.93\text{miles/second}$$

$$\approx 11.2\text{kilometers/second}$$

$$\approx \text{Mach 33}.$$  

- Here is a table of some other escape velocities, taken from the wikipedia (slightly modified). We assume we are at initially at rest at the given location. (Escaping from the sun starting on the orbiting Earth is easier and more complicated.)

<table>
<thead>
<tr>
<th>Location</th>
<th>with respect to</th>
<th>$v_0$(km/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Earth</td>
<td>Earth’s Gravity</td>
<td>11.2</td>
</tr>
<tr>
<td>Moon</td>
<td>Moon’s Gravity</td>
<td>2.38</td>
</tr>
<tr>
<td>Sun</td>
<td>Sun’s Gravity</td>
<td>617.5</td>
</tr>
<tr>
<td>Earth</td>
<td>Sun’s Gravity</td>
<td>42.1</td>
</tr>
<tr>
<td>Solar System</td>
<td>Milky Way</td>
<td>$\approx 500$</td>
</tr>
<tr>
<td>Event Horizon</td>
<td>Black Hole</td>
<td>Speed of Light</td>
</tr>
</tbody>
</table>
• Improper integrals occur in probability theory.

• A function $f$ is a **probability density function** if

\[ f(x) \geq 0, \quad -\infty < x < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} f(x) \, dx = 1. \]

• $\int_{a}^{b} f(x) \, dx$ is the **likelihood** or **probability** that the even $x$ is in the interval $[a, b]$.

• The **mean** of $f$ is

\[ \mu = \int_{-\infty}^{\infty} x f(x) \, dx \]

and the **variance** of $f$ is

\[ \sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) \, dx. \]

• $\sigma$ is called the **standard deviation**.

• For example, let $\lambda > 0$ be a parameter and let

\[ f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{else} \end{cases} \]

• Verify that $f$ is a probability density function, and compute its mean and variance.