Math 1220-3

Notes of 10/3/23

8.2 Other Indeterminate Forms

• Recall the Rule of L’Hôpital. Suppose

\[ \lim_{x \to u} f(x) = \lim_{x \to u} g(x) = 0 \]

and

\[ \lim_{x \to u} \frac{f'(x)}{g'(x)} \]

exists. Then

\[ \lim_{x \to u} \frac{f(x)}{g(x)} = \lim_{x \to u} \frac{f'(x)}{g'(x)}. \]

Numerator and Denominator must both go to zero.

We are differentiating numerator and denominator separately, we are not applying the quotient rule!

• The Rule of L’Hôpital tells us how to handle the indeterminate expression or indeterminate form

\[ \frac{0}{0} \]
• The other indeterminate forms are

\[ \frac{\infty}{\infty}, \ 0 \times \infty, \ \infty - \infty, \ 0^0, \ \infty^0, \ 1^\infty. \]

• In all of these cases we have two ingredients that approach the given limits and we ask what happens to the expression in the limit.

• All of these cases go back to the Rule of L’Hôpital, in one way or other.

• We can also consider limits that are infinite, or limits as \( x \) approaches \( +\infty \) or \( -\infty \), or that are one-sided.
We start with a fanciful example. Suppose you and Elon Musk\(^{-1}\) start a new job and stay at it forever. You make $40 per hour, EM makes $20 per hour, and you both work the same number of hours per week. What is the limit of the ratio of your and EM’s lifetime earnings as time goes to infinity?

\(^{-1}\) according to forbes.com currently (9/28/23) the richest person in the world with a net worth of $245.8 billion.
• In general, suppose
\[ \lim_{x \to u} |f(x)| = \lim_{x \to u} |g(x)| = \infty \]
and
\[ \lim_{x \to u} \frac{f'(x)}{g'(x)} \]
exists. Then
\[ \lim_{x \to u} \frac{f(x)}{g(x)} = \lim_{x \to u} \frac{f'(x)}{g'(x)}. \]

• We already did one example. Here is another: Suppose \( p \) is any polynomial. What is
\[ \lim_{x \to \infty} \frac{p(x)}{e^x} ? \]

• How about
\[ \lim_{x \to \infty} \frac{e^x}{p(x)} ? \]
$0 \times \infty$

- The limit can be obtained by converting the product into a quotient and applying the Rule of L’Hôpital.

- Example

\[
L = \lim_{x \to \frac{\pi}{2}^-} \tan x \ln \sin x
\]
\[ \infty - \infty \]

- Converting a difference to a quotient can be tricky!

- Example:

\[
L = \lim_{x \to 1^+} \left( \frac{x}{x - 1} - \frac{1}{\ln x} \right)
\]
0^0, \ \infty^0, \ 1^\infty

- Handle these expressions by computing the limit of the logarithm.

After computing that limit, **don't forget** to undo the logarithm!

- Example:

\[
L = \lim_{x \to 0^+} x^x
\]
What is the graph of $f(x) = x^x$, anyway?
Figure 1. The graph of $y = x^x$.
• A trick question: What about

\[ L = \lim_{x \to 0^+} (1 + x)^x \]
• More interesting is

\[ L = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \]

• This is the factor by which you multiply your investment each year if the interest rate is 100% and the investment is compounded continuously.

• Any guesses?
• Example 9, textbook.

$$\lim_{x \to 0^+} (\sin x)^{1/x}$$
• Not all limits are suitable for the Rule of L’Hôpital.

• Here are some as yet unfamiliar limits that we will study soon:

\[
\lim_{n \to \infty} \sum_{i=0}^{n} r^i = \frac{1}{1 - r}, \quad |r| < 1
\]

\[
\int_{0}^{1} \frac{1}{x^2} \, dx = \lim_{a \to 0^+} \int_{a}^{1} \frac{1}{x^2} \, dx = \infty
\]

\[
\int_{1}^{\infty} \frac{1}{x^2} \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{1}{x^2} \, dx = 1
\]

\[
\int_{-\infty}^{\infty} e^{-x^2} \, dx = 2 \lim_{b \to \infty} \int_{0}^{b} e^{-x^2} \, dx = \sqrt{\pi}
\]

• The last three limits are our next subject.

• We’ll get to the first kind after that.