Announcements

• No class Monday
• You get a message from me after every exam or home work.
• The next one tomorrow morning.

7.6 Techniques of Integration

• This is a catch all section. We discussed three major techniques:

1. Integration by Substitution. This is largely a bag of tricks. Based on the inverse process of the chain rule:

\[ \int f(g(x))g'(x)dx = F(g(x)) \]

where \( F \) is an antiderivative of \( f \): \( F' = f \). It usually takes the form of finding a suitable substitution \( u = g(x) \). For indefinite integrals, you need to return to the original variable, and don’t forget the integration constant. For definite integrals, you do not need to return to the original variable, but you do need to change the limits of integration to
their $u$ values. Definite integrals are numbers, there is no integration constant.

2. **Integration by Parts.** The reverse process of the product rule. You integrate a product, and you choose one factor to differentiate and the other to integrate. Integration by parts is a major technique with many applications in many contexts.

3. **Partial Fractions.** Let’s you integrate any proper rational function if you can factor the denominator.

- Let’s do some more examples.
Example 9, section 7.5, generalized. Logistic Population Growth.

\[ y' = ky(L - y), \quad 0 < y(0) = y_0 < L, \quad k > 0. \]

Figure 1. Logistic Growth, \( L = 1, y_0 = 0.01, k = 0.5. \)
\[ y' = ky(L-y) \quad y(0) = y_0 \]

\[ \frac{dy}{dt} = ky(L-y) \]

\[ \frac{dy}{y(L-y)} = k\,dt \]

\[ \frac{1}{y(L-y)} = \frac{A}{y} + \frac{B}{L-y} \]

\[ A(L-y) + By = 1 \]

\[ \begin{align*}
    y &= 0 \quad AL = 1 \quad A = \frac{1}{L} \\
    y &= L \quad BL = 1 \quad B = \frac{1}{L}
\end{align*} \]

\[ \left( \frac{1}{L} \frac{1}{y} + \frac{1}{L} \frac{1}{L-y} \right)\,dy = k\,dt \]

\[ \left( \frac{1}{y} + \frac{1}{L-y} \right)\,dy = kL\,dt \]

\[ \ln y - \ln(L-y) = kLt + C \]

\[ \ln \frac{y}{L-y} = kLt + C \]

\[ t=0 \quad y = y_0 \]

\[ \ln \frac{y_0}{L-y_0} = C \quad y = \frac{L}{2} \]
\[
\begin{align*}
\ln \frac{Y}{L-Y} &= kL+ \ln \frac{Y_0}{L-Y_0} \\
\frac{Y}{L-Y} &= e^{kL} + \frac{Y_0}{e^{kL}} \\
&= e^{kL} e^{\frac{Y_0}{L-Y_0}} \\
\frac{Y}{L-Y} &= \frac{Y_0}{L-Y_0} e^{kL} = R \\
\frac{Y}{L-Y} &= R \\
Y &= R(L-Y) \\
Y &= RL - RY \\
y + Ry &= RL \\
y(1+R) &= RL \\
\frac{Y}{1+R} &= \frac{RL}{1+R} \\
&= \frac{RL}{e^{kL}} \\
&= \frac{Y_0}{e^{kL}} e^{kL} \\
\end{align*}
\]

* multiply numerator and denominator with \((L-Y_0)e^{-kLt}\)
\[ Y(t) = \frac{L Y_0}{Y_0 + (L - Y_0) e^{-kt}} \]

\[ Y(0) = Y_0 \]

\[ \lim_{t \to \infty} Y(t) = L \]

\[ \gamma = e^{-kt} \]

exercise C.0)
\[ I = \int \frac{x^2}{x^2 - 1} \, dx = \int 1 + \frac{1}{x^2 - 1} \, dx \]

\[
\frac{1}{x^2 - 1} \sqrt{\frac{x^2}{x^2 - 1}} = 1
\]

\[ I + \frac{1}{x^2 - 1} = \frac{x^2 - 1 + 1}{x^2 - 1} = \frac{x^2}{x^2 - 1} \]

\[
\frac{1}{x^2 - 1} = \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}
\]

\[ A(x-1) + B(x+1) = 1 \]

\[ x=1 \quad 2B = 1 \quad B = \frac{1}{2} \]

\[ x=-1 \quad -2A = 1 \quad A = -\frac{1}{2} \]

\[ I = \int 1 + \frac{1}{2} \frac{1}{x-1} - \frac{1}{2} \frac{1}{x+1} \, dx \]

\[ = x + \frac{1}{2} \ln|x-1| - \frac{1}{2} \ln|x+1| + C \]
Example 8, page 407, textbook.

\[
I = \int \frac{6x^2 - 15x + 22}{(x + 3)(x^2 + 2)^2} \\
= \int \frac{A}{x + 3} + \frac{Bx + C}{x^2 + 2} + \frac{Dx + E}{(x^2 + 2)^2}
\]

• This gives the basic equation

\[
A(x^2+2)^2+(Bx+C)(x+3)(x^2+2)+(Dx+E)(x+3) = 6x^2-5x+22
\]

which can be solved (exercise!) to give

\[
A = 1, \quad B = -1, \quad C = 3, \quad D = -5, \quad E = 0
\]

• Our integral becomes

\[
I = \int \frac{1}{x + 3} + \frac{-x + 3}{x^2 + 2} - \frac{5x}{(x^2 + 2)^2} \, dx.
\]

• Let’s look at the integrals individually:
\[
\int \frac{5x}{(x^2+2)^2} \, dx = -\frac{5}{2} \cdot \frac{1}{x^2+2} = -\frac{5}{2(x^2+2)}
\]

\[
\frac{d}{dx} - \frac{5}{2} (x^2+2)^{-1} = \frac{5}{2} (x^2+2)^{-2} \cdot 2x
\]

\[
\int \frac{1}{x+3} = \ln |x+3|
\]

\[
\int \frac{-x+3}{x^2+2} = \int \frac{-x}{x^2+2} + \int \frac{3}{x^2+2}
\]

\[
\int \frac{-x}{x^2+2} \, dx = -\frac{1}{2} \ln |x^2+2|
\]

\[
\int \frac{3}{x^2+2} \, dx = \int \frac{3}{x^2} + \frac{3}{2}
\]
\[ \int \frac{3}{x^2 + 2} \, dx = \frac{1}{2} \int \frac{3}{\frac{x^2}{2} + 1} \, dx \]

\[ u^2 = \frac{x^2}{2} \]

\[ u = \frac{x}{\sqrt{2}} \]

\[ du = \frac{1}{\sqrt{2}} \, dx \]

\[ dx = \sqrt{2} \, du \]

\[ = \frac{1}{2} \int \frac{3}{u^2 + 1} \sqrt{2} \, du \]

\[ = \frac{\sqrt{2}}{2} \arctan u \]

\[ = \frac{3}{\sqrt{2}} \arctan \left( \frac{x}{\sqrt{2}} \right) \]
• putting it all together we get

\[
I = \int \left( \frac{1}{x + 3} + \frac{-x + 3}{x^2 + 2} - \frac{5x}{(x^2 + 2)^2} \right) dx
\]

\[
= \ln |x + 3| - \frac{1}{2} \ln(x^2 + 2) + \frac{3}{\sqrt{2}} \arctan \frac{x}{\sqrt{2}} + \frac{5}{2(x^2 + 2)} + C
\]

• Exercise: check by differentiation!

• We are done with chapter 7!