We could spend a lot of time and effort to develop a high level of expertise in integration by substitution, but that would not be a good use of our time given modern computer resources.

### 7.5 Partial Fractions

- The next topic is much more important.
- We’ll start today and do more examples on Friday.
- Partial Fractions is a technique to integrate rational functions.
- A rational function is the quotient of two polynomials.

A rational function is **proper** if the degree of the numerator is less than that of the denominator.

- Examples:

  \[
  f(x) = \frac{2}{(x + 1)^3}
  \]

  \[
  g(x) = \frac{2x + 2}{x^2 - 4x + 8}
  \]

  \[
  h(x) = \frac{x^5 + 2x^3 - x + 1}{x^3 + 5x}
  \]
Using long division, we can always convert an improper rational function into the sum of a polynomial and a proper rational function.

- Example (function \( h \) above)

\[
h(x) = \frac{x^5 + 2x^3 - x + 1}{x^3 + 5x}
\]

\[
\begin{array}{c|ccccc}
 & x^2 & -3 \\
\hline
x^3 + 5x & x^5 & +2x^3 & +0x^2 & -x & +1 \\
& x^5 & +5x^3 & & & \\
\hline
& -3x^3 & -15x & + & 14x & +1
\end{array}
\]

\[
h(x) = \frac{x^5 + 2x^3 - x + 1}{x^3 + 5x} = x^2 - 3 + \frac{14x + 1}{x^3 + 5x}
\]

\[
= \frac{(x^2 - 3)(x^3 + 5x) + 14x + 1}{x^3 + 5x}
\]

\[
= \frac{x^5 + 2x^3 - x + 1}{x^3 + 5x}
\]
• Of course we can integrate polynomials.

• So let’s focus on proper rational functions.

• Here is a summary of what we have learned so far. (We ignore integration constants today.)

\[
\int \frac{1}{x+a} \, dx = \ln |x+a| + C
\]

\[
\int \frac{x}{x^2+a} \, dx = \frac{1}{2} \ln |x^2+a|
\]

\[
\int \frac{1}{(x+a)^2} \, dx = \int (x+a)^{-2} \, dx = -(x+a)^{-1} = \frac{-1}{x+a}
\]

\[
\int \frac{1}{1+x^2} \, dx = \arctan x
\]
• Example: \[ I = \int \frac{3x+7}{x^2 + 5x + 6} \, dx \]

• Hint:

\[
\frac{3x + 7}{x^2 + 5x + 6} = \frac{1}{x + 2} + \frac{2}{x + 3} = \frac{x + 3 + 2(x + 2)}{(x + 2)(x + 3)}
\]

\[
\overline{I} = \int \frac{3x + 7}{x^2 + 5x + 6} \, dx
\]

\[
= \int \left( \frac{1}{x + 2} + \frac{2}{x + 3} \right) \, dx
\]

\[
= \ln |x + 2| + 2 \ln |x + 3|
\]

\[
= \ln \left| \frac{(x + 2)(x + 3)^2}{(x + 2)(x + 3)} \right|
\]

\[
\overline{I} = \frac{(x + 3)^2 + (x + 2)2(x + 3)}{(x + 2)(x + 3)^2}
\]

\[
= \frac{(x + 3) + 2(x + 2)}{(x + 2)(x + 3)}
\]

\[
= \frac{3x + 7}{x^2 + 5x + 6}
\]
• How do we come up with the partial fraction decomposition

\[ \frac{3x + 7}{x^2 + 5x + 6} = \frac{1}{x + 2} + \frac{2}{x + 3} \]

\[
\frac{3x + 7}{x^2 + 5x + 6} = \frac{3x + 7}{(x+2)(x+3)}
\]

\[
= \frac{A}{x+2} + \frac{B}{x+3}
\]

\[
= \frac{A(x+3) + B(x+2)}{(x+2)(x+3)}
\]

\[
A(x + 3) + B(x + 2) = 3x + 7
\]

\[
(A+B)x + (3A+2B) = 3x + 7
\]

\[
\begin{cases} 
A + B = 3 \\
3A + 2B = 7
\end{cases} \implies A = 1, \quad B = 2
\]

\[
3x + 7 = A(x+3) + B(x+2)
\]

\[
x = -3 \implies -2 = -B \implies B = 2
\]

\[
x = -2 \implies 1 = A
\]
Example 5, textbook

\[ I = \int \frac{x}{(x-3)^2} \, dx \]

\[ \frac{x}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2} = \frac{A+B}{x-3} = \frac{C}{x-3} \]

\[ \text{Nope} \]

\[ = \frac{A}{x-3} + \frac{B}{(x-3)^2} \]

\[ = \frac{A(x-3) + B}{(x-3)^2} \]

\[ = \frac{x}{(x-3)^2} \]

\[ A(x-3) + B = x \implies A = 1 \]

\[ x = 3 \quad B = 3 \]

\[ A(x-3) + 3 = x \quad A = 1 \]
\[ \int \frac{x}{(x-3)^2} \, dx = \int \left( \frac{1}{x-3} + \frac{3}{(x-3)^2} \right) \, dx \]

\[ = \ln|x-3| - \frac{3}{x-3} = I \]

\[ I' = \frac{1}{x-3} + \frac{3}{(x-3)^2} \]

\[ = \frac{x-3 + 3}{(x-3)^2} = \frac{x}{(x-3)^2} \]
\[ I = \int \frac{6x^2 - 3x + 1}{(4x+1)(x^2+1)} \, dx \]

\[
\int \frac{6x^2 - 3x + 1}{(4x+1)(x^2+1)} \, dx = \int \left( \frac{2}{4x+1} + \frac{x-1}{x^2+1} \right) \, dx
\]

\[
= \int \frac{2}{4x+1} + \int \frac{x}{x^2+1} - \int \frac{1}{x^2+1} \, dx
\]

\[
= \frac{1}{2} \ln(4x+1) + \frac{1}{2} \tan^{-1}(x) - \frac{1}{2} \ln(x^2+1)
\]

\[
\frac{6x^2 - 3x + 1}{(4x+1)(x^2+1)} = \frac{A}{4x+1} + \frac{Bx+C}{x^2+1}
\]

\[
A(4x+1) + (Bx+C)(x^2+1) = 6x^2 - 3x + 1
\]

\[
x = -\frac{1}{4} \quad A \frac{17}{16} = \frac{6}{16} + \frac{3}{4} + 1 = \frac{6+12+16}{16}
\]

\[
\frac{17}{16} A = \frac{34}{16}
\]

\[
A = 2
\]
\( x = 0 \quad A + c^1 = 1 \)
\( 2 + c^1 = 1 \quad c^1 = -1 \)

\( x = 1 \quad 2A + (B + c^1)5 = 4 \)
\( 41 + (B - 1)5 = 4 \)
\( (B-1)5 = 0 \)
\( B = 1 \)

\[ y' = ky(L - y), \quad 0 < y(0) = y_0 < L, \quad k > 0. \]