Recall integration by parts:

$$\int_a^b uv' \, dt = uv\bigg|_a^b - \int_a^b u'v \, dt.$$ 

We’ll look at another example of integration by parts which will lead to the concept of a **Taylor series** which we will approach from a different direction and study in great depth later in the semester.

For any non-negative integer $n$ we define $n!$ (*n*-factorial) by

$$n! = \begin{cases} 
1 & \text{if } n = 0 \\
1 \times 2 \times 3 \times \ldots \times n & \text{if } n > 1
\end{cases}$$

The following table gives the first few factorials:

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n!$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>24</td>
<td>120</td>
<td>720</td>
<td>5040</td>
<td>40,320</td>
</tr>
</tbody>
</table>

It turns out that under suitable assumptions

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$
• The right hand side of this equation is called a **Taylor Series**. More about this in a few weeks, today we’ll look at how to get this equation by integration by parts.
• Example:

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{24!} + \ldots \]
7.3 Trigonometric Integrals

• Integration by parts is a major concept and technique. By comparison, section 7.3 is a bag of tricks. Will mostly do a bunch of examples.

• Example 1: $I = \int \sin x \cos^2 x \, dx$

• Example 2: $I = \int \sin^3 x \, dx$
\[ I = \int \sin^5 x \, dx \]
\[ I = \int_{-\pi}^{\pi} \sin^2 x \, dx \]

\[ I = \int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} \, dx \]
\[ I = \int_{-\pi}^{\pi} \sin(nx) \sin(mx) \, dx \] where \( m \) and \( n \) are integers.
What about $\int_{-\pi}^{\pi} \sin^{13} x \, dx$?
• Important idea: **Reduction Formula**

• Define:

\[
I_n = \int_0^1 t^n e^t \, dt.
\]

Then

\[
I_0 = \int_0^1 e^t \, dt = e - 1.
\]

Moreover,

\[
I_n = \int_0^1 t^n e^t \, dt = t^n e^t \bigg|_0^1 - n \int_0^1 t^{n-1} e^t \, dt.
\]

Thus

\[
I_n = e - nI_{n-1}.
\]

• This is an example of a reduction formula. We can use it to compute \(I_1, I_2, I_3\), one after the other:
the expressions look complicated and unpredictable, but something (perhaps) surprising becomes apparent when they are listed numerically:

\[ I_1 = 1 = 1.000 \]
\[ I_2 = e - 2 = 0.718 \]
\[ I_3 = -2e + 6 = 0.563 \]
\[ I_4 = 9e - 24 = 0.465 \]
\[ I_5 = -44e + 120 = 0.396 \]

In hindsight, of course the integral is always positive, and goes to zero as \( n \) goes to infinity.