Math 1220-3

- Study Session (Andy) 10:45, JWB 140
- Office hours 11:50
- graded exam should be accessible on Canvas
7.3 Trigonometric Integrals

- Integration by parts is a major concept and technique. By comparison, section 7.3 is a bag of tricks. Will mostly do a bunch of examples.

- Example 1: \( I = \int \sin x \cos^2 x \, dx \)

- Example 2: \( I = \int \sin^3 x \, dx \)
\[ I = \int \sin^5 x \, dx \]
\[ I = \int_{-\pi}^{\pi} \sin^2 x \, dx \]

\[ I = \int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} \, dx \]
Figure 1. Graphs of $\sin x$ and $\sin^2 x$. ..
\[ I = \int_{-\pi}^{\pi} \sin(nx) \sin(mx) \, dx \] where \( m \) and \( n \) are integers.
What about \( \int_{-\pi}^{\pi} \sin^{13} x \, dx \)?
• Important idea: **Reduction Formula**

• Define:

\[ I_n = \int_0^1 t^n e^t \, dt. \]

Then

\[ I_0 = \int_0^1 e^t \, dt = e - 1. \]

Moreover,

\[ I_n = \int_0^1 t^n e^t \, dt = t^n e^t \bigg|_0^1 - n \int_0^1 t^{n-1} e^t \, dt. \]

Thus

\[ I_n = e - nI_{n-1}. \]

• This is an example of a reduction formula. We can use it to compute \( I_1, I_2, I_3 \), one after the other:
the expressions look complicated and unpredictable, but something (perhaps) surprising becomes apparent when they are listed numerically:

\[
\begin{align*}
I_0 &= e - 1 &= 1.718 \\
I_1 &= 1 &= 1.000 \\
I_2 &= e - 2 &= 0.718 \\
I_3 &= -2e + 6 &= 0.563 \\
I_4 &= 9e - 24 &= 0.465 \\
I_5 &= -44e + 120 &= 0.396
\end{align*}
\]

In hindsight, of course the integral is always positive, and goes to zero as \(n\) goes to infinity.
• A fancy application of Integration by Parts

\[ \int f'g = fg - \int fg' \]

• Recall that the factorial \( n! \) is defined by

\[ 0! = 1! = 1, \quad n! = n \times (n-1) \times (n-2) \times \ldots \times 1. \]

• The first few factorials are

\[
\begin{array}{cccccccccccc}
  n & : & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
  n! & : & 1 & 1 & 2 & 6 & 24 & 120 & 720 & 5,040 & 40,320 \\
\end{array}
\]

• If time allows use integration by parts to derive the MacLaurin formula:

\[
f(x) = f(0) + xf'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{6} f'''(0) + \frac{x^4}{4} f^{(4)}(0) + \ldots \\
= \sum_{i=0}^{\infty} \frac{x^i}{i!} f^{(i)}(0)
\]

• For example, if

\[ f(x) = e^x, \quad f^{(i)}(x) = e^x, \quad \text{and} \quad f^{(i)}(0) = 1, \]

then

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots = \sum_{i=0}^{\infty} \frac{x^i}{i!}. \]

• This is actually a good way to compute the exponential!