Math 1220-3

Notes of 9/18/23

Summary of Chapter 6

As you know, our first midterm exam will take place on Friday, September 22, at our regular class time. It will be on Chapter 6. Note, however, that integration by substitution, which is the focus of section 7.1, actually already occurred in Math 1210, way back in section 3.8, and thus is a familiar technique. You may find integration by substitution appropriate for one or two problems on the exam.

We will spend the whole week prior to the exam on a review and practicing some techniques.

Since this is our first exam, here is a transcript of the instructions on the cover page of the exam. (The font size in the actual exam will be the usual 12pt, much smaller than the font size in these notes.)

Instructions

0. This exam will be processed by Gradescope. This means that grading will be more informative for you, and more consistent. But you should make some extra effort to make grade-scope effective. Please use a writing utensil that draws a dark curve (not a hard pencil, for example). For each problem, please enter
your final answer, **and nothing else**, in the box(es) provided with the question. Everywhere, but particularly in those boxes, make an effort to write clearly and avoid crossing out things if you can. Write all work in the remaining space provided with the problem. If you write a note or comment for me in a random spot I may miss it. Therefore, at the end of the exam there is an extra large box that you can use for additional work or for notes to me. Most people will leave that box blank. However, I will have Gradescope treat those box as an ungraded exam questions. That way, gradescope will make me look at the box if it is not left blank.

1. This exam is **closed books and notes, no electronic devices, no scratch paper**.

2. Use these sheets to record your work and your results. Use the space provided. **Show all work**.

3. Please note: To avoid disruption and distraction I won’t be able to answer questions during the exam. If you believe there is a mistake in one of the problems write down an appropriate note and if you are right you will receive generous credit.

4. The questions on this test are deliberately simple. Also, you have seen all of these questions before, in class, or on the home works! You should not be rushed and have time to answer all questions carefully and check your
answers. **Accuracy is more important than speed.** Don’t get stuck on any one problem. If you can’t answer a question immediately go on and return to that question only after you have answered the others.

5. Simplify any algebraic expressions and reduce any fractions.

6. You don’t need to approximate mathematical expressions with decimal expressions.

7. Do not omit the integration constant for indefinite integrals.

8. If you are done before the allotted time is up I recommend that you stay and use the remaining time to check your answers.

9. All questions have equal weight.

**Be Prepared**

- The exam is focused on the application, manipulation, properties, derivatives, and integrals of functions involving exponentials and logarithms, and inverse trig functions. The key to doing well on any exam is to understand the subject. Focusing on a narrow list of anticipated questions is counterproductive. However, as discussed at the beginning of the semester, the exam questions will be taken straight from the home works (hw 1–4) and the class notes for chapter 6 (August 25 through September 12). I may simplify the problem a bit, take only part of a problem, or change
some numbers. (Many ww problems have different numbers for different people anyway.)

To be well prepared for the exam you want to understand everything we did in class, and you want to understand how you computed the answers to the home work problems. It’s not enough just to have gotten credit for your answer.

Topics

• Following is a review of the subject we covered in chapter 6. The list is neither self contained nor complete. Each item should trigger your memory and understanding of the surrounding subject. If it does not then go back over your notes, the textbook, and past home work problems, and make sure it does.

• Definition of the natural logarithm:

\[ \ln x = \int_{1}^{x} \frac{1}{t} \, dt \]  

(1)

• Properties of the natural logarithm (same as for any logarithm):
\[
\ln(uv) = \ln u + \ln v, \\
\ln\left(\frac{u}{v}\right) = \ln u - \ln v, \\
\ln(1) = 0, \\
\ln(u^v) = v \ln u.
\]

- The derivative of the natural logarithm, by its definition and the fundamental theorem of calculus:

\[
\frac{d}{dx} \ln x = \frac{1}{x} \quad (3)
\]

- The natural exponential is the inverse of the natural logarithm:

\[
\exp(\ln x) = x, \quad x > 0 \\
\ln \exp(x) = x, \quad -\infty < x < \infty 
\]

\[
\exp(t) = e^t, \quad e = 2.71828\ldots \quad (5)
\]

- Be clear in your mind about domain and range of exponential and logarithmic functions.

- The exponential equals its own derivative:

\[
\frac{d}{dt} e^t = e^t. \quad (6)
\]
• It has the standard properties:

\[ e^{u+v} = e^u e^v, \quad (e^u)^v = e^{uv}, \quad e^0 = 1. \quad (7) \]

• A more general exponential is

\[ f(t) = Ca^t = Ce^{kt} \quad \text{where} \quad k = \ln a. \]

\[ a^t = (e^{\ln a})^t = e^{(\ln a)t} \]

• Key properties of exponentials include:

  – The derivative is proportional to the function:

\[ \frac{d}{dt} f(t) = \frac{d}{dt} Ce^{kt} = kCe^{kt} = kf(t). \]

  – Over a time interval of a given length the function is multiplied with a constant factor which is independent of the beginning of the time interval.

\[ f(t+T) = Ce^{k(t+T)} = e^{kT} Ce^{kt} = e^{kT} f(t). \]

  – In more colloquial terms: a function grows exponentially if it grows by a fixed percentage over any time interval of a given length.

  – Of course, the factor by which we multiply may be smaller than 1, in which case we talk about exponential decay.

• Of course you should be able to do anything with exponentials and logarithms that you learned
in preCalculus. This includes simplifying logarithmic and exponential expressions, and solving logarithmic or exponential equations.

- **Doubling time, half life.** You should be able to construct exponential functions with given initial population, and given half lives or doubling times. You should also be able to compute the time at which the population is multiplied with a given factor.

- **Integration by substitution** is the inverse process of the chain rule. Thus, if $F' = f$ then

$$
\int f(g(x))g'(x) \, dx = F(g(x)) + C. \quad (8)
$$

You may, or may not, use the explicit substitution $u = g(x)$. For indefinite integrals, we usually go back to an expression in $x$, for definitive integrals we do not, but then we have to change the limits of integration. We have seen many examples where the substitution in (8) is obvious, but sometimes it is not.

- **Logarithmic differentiation.** Take the logarithm first, then differentiate implicitly and solve for the derivative of interest.

- **Differentiation of inverse functions:** Start with

$$
f(f^{-1}(x)) = x, \quad (9)
$$

differentiate implicitly, and solve for the derivative of $f^{-1}$. It’s better to use and understand
the process, but you get the formula

\[(f^{-1})' (y) = \frac{1}{f'(x)} \quad \text{where} \quad y = f(x).\]  

\[(10)\]

- In particular we applied this idea to obtain the derivative of the exponential, and the derivatives of some inverse trig functions. As mentioned, the exponential equals its own derivative. The most important formula for the inverse trig functions is

\[\frac{d}{dx} \arctan(x) = \frac{1}{1 + x^2}. \]  

\[(11)\]

- We also saw that

\[\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1 - x^2}}\]

\[\frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1 - x^2}} = - \frac{d}{dx} \arcsin x \]

\[(12)\]

- The hyperbolic sine, cosine, and tangent are defined by

\[\sinh t = \frac{e^t - e^{-t}}{2},\]

\[\cosh t = \frac{e^t + e^{-t}}{2}, \quad \text{and} \]

\[\tanh t = \frac{\sinh t}{\cosh t}.\]  

\[(13)\]
• They are called *hyperbolic* because

\[ \cosh^2 t - \sinh^2 t = 1 \]  

(14)

and the graph of \( x^2 - y^2 = 1 \) is a hyperbola. They are called *sine* and *cosine* because (14) is similar to the fundamental property

\[ \cos^2 t + \sin^2 t = 1 \]  

(15)

of the trigonometric functions.

• Their derivatives are given by

\[
\begin{align*}
D_t \sinh t &= \cosh t, \\
D_t \cosh t &= \sinh t, \\
D_t \tanh t &= \frac{1}{\cosh^2 t}.
\end{align*}
\]

(16)

• We also computed expressions for the inverse of the hyperbolic sine and its derivative.

• **Separable differential equations** are of the form

\[
\frac{dy}{dx} = f(x)g(y). \tag{17}
\]

Separate the variables, integrate on both sides, and solve for the function you want.

• **First order linear differential equations** are of the form

\[
y' + P(x)y = Q(x). \tag{18}
\]
The integrating factor

\[ e^{\int P(x) \, dx} \quad (19) \]

is the exponential of an antiderivative of the factor of the dependent variable. Multiply with the integrating factor, integrate on both sides, and solve for the function you want.

- Differential equations are important because they can be used to model natural processes.

- Most differential equations cannot be solved analytically, and need to be solved approximately. We discussed just one numerical method, Euler’s Method, which is the tip of a very large ice-berg.

**Reminder: Differentiation Rules**

- Part of what we accomplished so far is an expansion of our repertoire of differentiation rules. For reference and as a reminder, here is a list of the rules we knew before starting Math 1220.
\[
\frac{d}{dx} x^r = r x^{r-1} \quad \text{Power Rule}
\]
\[
(f + g)' = f' + g' \quad \text{Sum Rule}
\]
\[
(f - g)' = f' - g' \quad \text{Difference Rule}
\]
\[
(kf)' = kf' \quad \text{Constant Multiple Rule}
\]
\[
\frac{d}{dx} \sin x = \cos x \quad \text{Sine Rule}
\]
\[
\frac{d}{dx} \cos x = -\sin x \quad \text{Cosine Rule}
\]
\[
(uv)' = u'v + uv' \quad \text{Product Rule}
\]
\[
\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} \quad \text{Quotient Rule}
\]
\[
\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) \quad \text{Chain Rule}
\]
Here is a list of rules we have so far learned in this semester:

<table>
<thead>
<tr>
<th>Rule</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Log Rule</td>
<td>( \frac{d}{dx} \ln x = \frac{1}{x} )</td>
</tr>
<tr>
<td>Exponential Rule</td>
<td>( \frac{d}{dx} e^x = e^x )</td>
</tr>
<tr>
<td>sinh Rule</td>
<td>( \frac{d}{dx} \sinh x = \cosh x )</td>
</tr>
<tr>
<td>cosh Rule</td>
<td>( \frac{d}{dx} \cosh x = \sinh x )</td>
</tr>
<tr>
<td>Inverse sin Rule</td>
<td>( \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}} )</td>
</tr>
<tr>
<td>Inverse cos Rule</td>
<td>( \frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1-x^2}} )</td>
</tr>
<tr>
<td>Inverse tan Rule</td>
<td>( \frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2} )</td>
</tr>
<tr>
<td>Inverse sinh Rule</td>
<td>( \frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{x^2+1}} )</td>
</tr>
</tbody>
</table>

\( \frac{1}{\tan x} \neq \tan^{-1} x = \arctan x = \text{atan } x \)
Exercises

• What is the general solution of \( y' = ky \)?

\[
\begin{align*}
\ln y &= kx + C \\
y &= ce^{kx} \\
2 &= e^{\ln 2}
\end{align*}
\]

• Compute \( \frac{d}{dx}2^x = \frac{d}{dx}(e^{\ln 2^x}) = \frac{d}{dx}e^{x\ln 2} = e^{x\ln 2} \cdot \ln 2 = (\ln 2)2^x \)

• Compute \( \int 2^x \, dx = \frac{2^x}{\ln 2} + C \)

• Express \( \log_2(x) \) in terms of the natural logarithm.

\[
\text{You invest money at 1 percent monthly interest. Find the doubling time of your investment, expressed in years.}
\]

\[
(1+ \frac{1}{100})^{12 \cdot t} = 2 \\
12t \cdot \ln(1.01) = \ln 2 \\
t = \frac{\ln 2}{12 \cdot \ln(1.01)}
\]

• Solve the differential equation \( y' = 2y \) where \( y(0) = 3 \).

\[
Y(t) = 3 \cdot e^{2t}
\]

\[
\text{CBD: } Y'(t) = 3 \cdot 2e^{2t} \cdot 2 = 2Y
\]
\[ f(x) = x^4 \quad f'(x) = 4x^3 \]

\[ x^4 = 81 \quad \therefore x = 3 = \sqrt[4]{81} \]

\[ f''(x) = 12x^2 \quad f'''(x) = 24x \quad f^{(4)}(x) = 24 \quad f^{(5)}(x) = 0 \]

\[ y = 4^x \]

\[ g(x) = 4^x \quad g'(x) = (\ln 4) \cdot 4^x \quad g''(x) = (\ln 4)^2 \cdot 4^x \]

\[ g^{(n)} = (\ln 4)^n \cdot 4^x \]

\[ \lim_{x \to \infty} \frac{x^4}{e^x} = 0 \]
• Suppose \( y = x^x \). Find \( y' \) in two different ways.

\[
\ln y = x \ln x \\
\frac{y'}{y} = \ln x + x \frac{1}{x} \\
y' = \left(1 + \ln x\right) x^x \\
x = e^{\ln x} \quad y = (e^{\ln x})^x = e^{x \ln x} \\
y' = e^{x \ln x} \cdot \ln x \cdot x^x \\
\]

\[ \frac{d}{dx} e^{\sin x} = e^{\sin x} \cdot \cos x \]

\[ \frac{d}{dx} (\ln x) e^{\sin x} = \frac{1}{x} e^{\sin x} + (\ln x) e^{\sin x} \cdot \cos x \]

• Do it two ways: What is \( \frac{d}{dx} e^{\ln x} \)?

\[
\frac{d}{dx} e^{\ln x} = e^{\ln x} \cdot \frac{1}{x} = \frac{x}{x} = 1
\]

\[ \frac{d}{dx} \ln(x^2 + x) = \frac{2x + 1}{x^2 + x} \]
\[ u = x^2 \quad \frac{du}{dx} = 2x \quad du = 2x\,dx \quad x\,dx = \frac{1}{2}\,du \]

\[ \int \frac{-1}{x(\ln x)^3} \, dx = \int \frac{1}{2(\ln x)^2} \, dx \]

\[ u = \ln x \quad du = \frac{1}{x} \, dx \quad \frac{1}{2} \ln^2 x + C \]

\[ \int xe^{x^2} \, dx = \frac{1}{2} e^{x^2} + C \]

\[ u = x^2 \quad du = 2x\,dx \quad x\,dx = \frac{1}{2}\,du \quad \int = \int \frac{1}{2} e^{u} \, du = \frac{1}{2} e^{u^2} + C \]

\[ \frac{d}{dx} e^{\frac{x}{1+x^2}} = e^{\frac{x}{1+x^2}} \frac{1 + 2x^2 - x \cdot 2x}{(1+x^2)^2} \]

\[ \frac{d}{dx} \arctan(\sinh x) = \frac{\cosh x}{1 + \sinh^2 x} = \frac{1}{\cosh x} \]

\[ \frac{d}{dx} (\arctan(x)) \sinh x = \frac{1}{1+x^2} \sinh x + \arctan x \cdot \cosh x \]

\[ \frac{d}{dx} \sinh x^2 = (\cosh x^2) 2x \]

\[ \frac{d}{dx} \sinh^2 x = 2 \sinh x \cosh x \]
The region bounded by \( e^{1-x^2}, y = 0, x = 0, \) and \( x = 1 \) is revolved about the \( y \)-axis. Find the volume of the resulting solid.

\[
V = \pi \int_0^1 2\pi x e^{1-x^2} \, dx
\]

\[
= -\frac{1}{2} - 2\pi \left[ e^{1-x^2} \right]_0^1
\]

\[
= -\pi \left( 1 - e \right) = \pi (e - 1)
\]
9/20/23

- sign error yesterday

\[ \int \frac{-1}{x(\ln x)^3} \, dx = \int \frac{-1}{x \ln^3 x} \, dx \]

study session today, 10:45 STB 140
hw 4 closes today
office hours 11:50 JWB 127
• Give a function with a doubling time of 3 days and an initial population of 1000 individuals

\[ f(t) = 1000 \cdot 2^{t/3} \]

• Give a function with a half life of 3 days and an initial weight of 1000 pounds.
• Find the solution of the differential equation

\[ y' = y - 1 \]

that satisfies the condition

\[ y(0) = 0. \]
Compute

\[ \int \frac{1}{x^2 + 2x + 2} \, dx = \]

\[ \int \frac{7}{x^2 - 6x + 25} \, dx = \]

\[ \int \frac{2x + 1}{x^2 + 1} \, dx = \]
Questions?