No office hours today, I am sorry.

7.2 Integration by Parts

- Integration by parts is the reverse process of the product rule.
  
  \[(uv)' = u'v + uv'\]

  \[uv' = (uv)' - u'v\]

  \[\int uv' = uv - \int u'v\]

  \[\int_a^b uv' = uv \bigg|_a^b - \int_a^b u'v\]
Of course, we can interchange the roles of \( u \) and \( v \). This gives the alternative formulas

\[
\begin{align*}
   u'v &= (uv)' - uv' \\
   \int u'v &= uv - \int uv' \\
   \int_a^b u'v &= uv \bigg|_a^b - \int_a^b uv'
\end{align*}
\]

- In more detailed notation we get the rules

\[
\int u(x)v'(x) \, dx = u(x)v(x) - \int u'(x)v(x) \, dx
\]

for indefinite integrals, and

\[
\int_a^b u(x)v'(x) \, dx = u(x)v(x) \bigg|_a^b - \int_a^b u'(x)v(x) \, dx
\]

for definite integrals.
• Another widely used but very confusing notation is obtained as follows: Rewrite

\[ \int uv' = uv - \int u'v \]

as

\[ \int u \frac{dv}{dx} dx = uv - \int \frac{du}{dx} v dx \]

and “cancel the dx” to get

\[ \int u dv = uv - \int v du. \]

• This looks like you are integrating \( u \) with respect to \( v \) and \( v \) with respect to \( u \), which makes no sense\(^{-1}\). I recommend you avoid that notation. If you encounter it simply multiply with \( \frac{dx}{dx} \) to get

\[ \int u dv \frac{dx}{dx} = uv - \int v du \frac{dx}{dx} \]

which can then be rewritten as

\[ \int u \frac{dv}{dx} dx = uv - \int \frac{du}{dx} v dx. \]

\(^{-1}\) In my humble opinion this is about as egregious as using a superscript -1 to indicate an inverse function.
Examples

• Recall

\[ \int uv' = uv - \int u'v \]

• The rest of today will consist of examples.

\[ \text{Notice the pattern: You integrate a product and you differentiate one factor and integrate the other! Make the choice of which is which that will simplify the integral.} \]

• To focus on the integration technique let us ignore integration constants today.
As a quick reminder here is again the last thing we did on Wednesday:

\[ I = \int xe^x \, dx = uv - \int u'v \]

\[ = xe^x - \int e^x \, dx \]

\[ = xe^x - e^x + C \]

\[ \therefore I = e^x + xe^x - e^x = xe^x \]
\[ I = \int x \sin x \, dx = uv - \int v \, du \]

\[ = -x \cos x + \int \cos x \, dx \]

\[ = -x \cos x + \sin x \]

\[ I = -\cos x + x \sin x + C \]
\[ I = \int x^2 \cos x \, dx = \text{uv} - \int \text{v} \, \text{du} \]

\[ = x^2 \sin x - 2 \int x \sin x \, dx \]

\[ = x^2 \sin x - 2 \left( -x \cos x + \sin x \right) \]

\[ = 2x \sin x + x^2 \cos x - 2 \left( -\cos x + x \sin x + \cos x \right) \]
\[ I = \int \ln x \, dx = uv - \int v \, du, \]
\[ = x \ln x - \int x \cdot \frac{1}{x} \, dx \]
\[ = x \ln x - x \]
\[ I = \ln x + x - 1 \]
\[ I = \int e^{kt} \sin t \, dt = uv - \int u'\, v \]

\[ = -e^{kt} \cos t + k \int e^{kt} \cos t \, dt \]

\[ = -e^{kt} \cos t + k \left[ uv - \int u'\, v \right] \]

\[ I = -e^{kt} \cos t + k \left[ e^{kt} \sin t - k e^{kt} \sin t \, dt \right] \]

\[ I = -e^{kt} \left( \cos t - k \sin t \right) - k^2 e^{kt} \sin t \, dt \]

\[ I = -e^{kt} \left( \cos t - k \sin t \right) - k^2 I \mid \]

\[ (1+k^2) I = -e^{kt} \left( \cos t - k \sin t \right) \]

\[ I = \frac{-e^{kt} \left( \cos t - k \sin t \right)}{1+k^2} \]
• Revisit the electric circuit problem Andy discussed on 9/5, except that this time instead of direct current from a battery we have alternating current from a power source. Thus we have a voltage $E$, a resistor $R$, and an inductor $L$, all in series, and the equation

$$LI' + RI = E = \sin t, \quad I(0) = 0$$

\[ \text{Diagram of circuit} \]

• Let $I(t)$ be the current at time $t$, as before, and suppose the applied voltage is given by

$$E(t) = \sin t$$

• We get the initial value problem

$$L \frac{dI}{dt} + RI = E(t) = \sin t, \quad I(0) = 0.$$

• **Hint:** Recall that

$$\int e^{kt} \sin t \, dt = \frac{1}{k^2 + 1} e^{kt} (k \sin t - \cos t) + C$$

$$I' + \frac{R}{L} I = \frac{1}{L} \sin t, \quad k = \frac{R}{L}$$
\[ I + kI = \frac{1}{L} \sin t \quad I \cdot e^{kt} \]

\[ I e^{kt} + kI e^{kt} = \frac{1}{L} e^{kt} \sin t \quad I \]

\[ I e^{kt} = \frac{1}{L} \cdot \frac{e^{kt}}{1+k^2} (ksint - cost) \quad + \frac{C}{L(1+k^2)} \]

\[ t=0 \quad I=0 \quad \Rightarrow \quad 0 = \frac{-1}{L(1+k^2)} + C \]

\[ C = \frac{1}{L(1+k^2)} \]

\[ I = \frac{1}{L(1+k^2)} (ksint - cost) + \frac{1}{L(1+k^2)} e^{-kt} \]

\[ = \frac{1}{L(1+k^2)} (ksint - cost + e^{-kt}) \]
Problem 84, p. 393 textbook. Show that

\[ I = \int_0^x \left( \int_0^t f(z) \, dz \right) \, dt = \int_0^x f(t)(x - t) \, dt. \]