Math 1220-3

Notes of 9/13/23

• Remember Study Session today, 10:45, JTB 140. Office Hours 11:50, JWB 127

• We are done with Chapter 6.

• Exam 1 on Chapter 6 will take place next week on Friday.

• The last home work covering Chapter 6 is hw 4. It will close next week on Wednesday before the exam.

• Remember that exam questions will be taken from home works and class notes.

• We’ll spend all of next week on review.

• In the mean time, for the remainder of this week let’s go on with Chapter 7.
Chapter 7, Integration Techniques

7.1 Basic Integration Techniques

• Every differentiation rule also is an integration rule. Just read it in the opposite direction. The derivative of $x^2$ is $2x$ and so $x^2$ is an antiderivative (or an indefinite integral) of $2x$.

• Antiderivatives are determined only up to a constant.

• Put differently, any integrable function has infinitely many antiderivatives.

• Integrals come in two flavors:
  – Indefinite integrals:
    \[
    \int f(x)\,dx = F(x) + C
    \]
    where $F' = f$, and
  – Definite integrals
    \[
    \int_{a}^{b} f(x)\,dx = F(b) - F(a)
    \]
    where, again, $F' = f$.

\[\text{a definite integral does not have an integration constant.}\]
a definite integral does not depend on the integration variable

an indefinite integral is only determined up to a constant.

- Following is a (partial) list of integration rules that we have seen in the past, most of them many times. To reduce clutter let’s ignore the integration constant (and any limits of integration).
\[ \int k \, dx = k \int + C \]
\[ \int x^r \, dx = \frac{x^{r+1}}{r+1}, \quad (r \neq -1) \]
\[ \int \frac{1}{x} \, dx = \ln |x| \]
\[ \int e^x \, dx = e^x \]
\[ \int f(x) \pm g(x) \, dx = \int f(x) \, dx \pm \int g(x) \, dx \]
\[ \int k \cdot f(x) \, dx = k \int f(x) \, dx \]
\[ \int \sin x \, dx = -\cos x \, dx \]
\[ \int \cos x \, dx = \sin x \, dx \]
\[ \int \tan x \, dx = \ln |\cos x| \]
\[ \int \frac{1}{1 + x^2} \, dx = \arctan x \]
\[ \int \frac{1}{\sqrt{1 - x^2}} \, dx = \arcsin x \]
\[ \int \frac{-1}{\sqrt{1 - x^2}} \, dx = \arccos x \]
\[ \int \sinh x \, dx = \cosh x \]
\[ \int \cosh x \, dx = \sinh x \]
The one main integration technique we learned in 1210 was integration by substitution. It is also the main topic of section 7.1.

Example:

\[ I = \int 2x \sin x^2 \, dx = -\cos x^2 + C \]

\[ \frac{d}{dx}(-\cos x^2) = + (\sin x^2) \cdot 2x = 2x \sin x^2 \]

Let \( u = x^2 \)

\[ \frac{du}{dx} = 2x \]

\[ du = 2x \, dx \]

\[ I = \int \sin u \, du = -\cos u + C = -\cos x^2 + C \]
• For indefinite integrals the substitution is just a tool on the way to the answer which will be in terms of the original integration variable.

For definite integrals we usually do not go back to the original variable, but **we must change the limits of integration.**

• For example:

\[
I = \int_0^{\sqrt{\pi}} 2x \sin x^2 \, dx = \int_0^\pi \sin u \, du = \left[ -\cos u \right]_0^\pi = 2 - \cos \pi + \cos 0 = 2
\]

\[
I = \left[ -\cos x^2 \right]_0^{\sqrt{\pi}} = -\cos \pi + \cos 0 = 2
\]
\[ \int \frac{x}{\cos^2 x^2} \, dx = \frac{1}{2} \int \frac{1}{\cos^2 u} \, du = \frac{1}{2} \tan u = \frac{1}{2} \tan x^2 \]

\( u = x^2 \)

\( du = 2x \, dx \)

\[ dx = \frac{1}{2} \, du \]

CBD

\[ \frac{d}{dx} \left( \frac{1}{2} \tan x^2 \right) = \frac{2x}{2 \cos^2 x^2} = \frac{x}{\cos^2 x^2} \]
\[ \frac{d}{dx} \frac{1}{x} = \frac{d}{dx} x^{-1} = -1 \cdot x^{-2} = \frac{-1}{x^2} \]

\[ \int \frac{6e^{1/x}}{x^2} \, dx = -6 \int e^u \, du = -6e^u + C = -6e^{1/x} \]

\[ u = \frac{1}{x} \]

\[ \frac{du}{dx} = \frac{1}{x^2} \]

\[ \frac{dx}{x^2} = -du \]
\[ \int_2^5 t \sqrt{t^2 - 4} \, dt = \frac{1}{2} \int_4^{25} \sqrt{u-4} \, du = \frac{1}{2} \int_4^{25} (u-4)^{1/2} \, du \]

\[ u = t^2 \]
\[ du = 2t \, dt \]
\[ t \, dt = \frac{1}{2} \, du \]

\[ = \frac{1}{3} \cdot 21^{3/2} = \text{Value} \]
\[ I = \int x^3(x^4 + 1)^{5/2} \, dx = \frac{1}{4} \cdot \frac{2}{7} (x^4 + 1)^{7/2} + C \]

\[ U = x^4 + 1 \]
\[ dU = 4x^3 \, dx \]
\[ x^3 \, dx = \frac{1}{4} \, dU \]

\[ I = \frac{1}{4} \int U^{5/2} \, dU \]
\[ = \frac{1}{4} \cdot U^{7/2} \cdot \frac{2}{7} = \frac{1}{4} \cdot \frac{2}{7} \cdot (x^4 + 1)^{7/2} + C \]
\[
\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \int \frac{1}{a \sqrt{1-(\frac{x}{a})^2}} \, dx = \frac{1}{a} \int \frac{1}{\sqrt{1-u^2}} \, du
\]

\[a^2-x^2 = a^2 \left(1 - \left(\frac{x}{a}\right)^2\right)\]

\[u = \frac{x}{a}\]

\[du = \frac{dx}{a}\]

\[dx = a \, du\]

\[= \frac{1}{a} \int \frac{1}{\sqrt{1-u^2}} \, du = \arcsin \frac{u}{a} + C\]

\[= \arcsin \frac{x}{a} + C\]
7.2 Integration by Parts

• A very major topic!

• Based on the **Product Rule**

\[(fg)' = f'g + fg'\]  

implies

\[fg = \int f'g + fg'\]

• That formula is not very useful. However, subtracting \(fg'\) in (1) gives

\[f'g = (fg)' - fg'\]

which leads to

\[
\int f'g = fg - \int fg' \]

and its definite integral version

\[
\int_a^b f'g = fg\big|_a^b - \int_a^b fg' \]

• The last two formulas are important and have lots of applications.
• You may find it clearer to write them with explicitly specified integration variables, as in

\[ \int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx \]

and its definite integral version

\[ \int_a^b f'(x)g(x)dx = f(x)g(x)|_a^b - \int_a^b f(x)g'(x)dx \]

• Also, of course, you may interchange the roles of \( f \) and \( g \).
\[
\int fg = fg - \int fg'
\]

- Example:

\[
I = \int x e^x \, dx = x e^x - \int e^x \, dx
\]

\[
= x e^x - e^x + C
\]

CBD \quad \frac{d}{dx} I = e^x + xe^x - e^x = xe^x