Math 1220-3

Notes of 1/31/18

• Remember: Tomorrow, Thursday, 9:40-10:30, here in JWB 335, Questions and Answers.

• We are done with Chapter 6.

• Exam on Chapter 6 will take place next week on Friday.

• The last home work covering Chapter 6 is hw 4. It will close on the exam day one minute before midnight.

⚠️ But you definitely want to be done with it well before the exam!

• Remember that exam questions will be taken from home works and class notes.

• We’ll spend all of next week on review.

• In the mean time, for the remainder of this week let’s go on with Chapter 7.
Chapter 7, Integration Techniques

7.1 Basic Integration Techniques

- Following is a (partial) list of integration rules that we have seen in the past, most of them many times. To reduce clutter let’s ignore the integration constant (and any limits of integration).
\[ \int k \, dx = k \, x \]
\[ \int x^r \, dx = \frac{x^{r+1}}{r+1}, \quad (r \neq -1) \]
\[ \int \frac{1}{x} \, dx = \ln|x| \]
\[ \int e^x \, dx = e^x \]
\[ \int f(x) \pm g(x) \, dx = \int f(x) \, dx \pm \int g(x) \, dx \]
\[ \int k \cdot f(x) \, dx = k \int f(x) \, dx \]
\[ \int \sin x \, dx = - \cos x \, dx \]
\[ \int \cos x \, dx = \sin x \, dx \]
\[ \int \tan x \, dx = \ln|\cos x| \]
\[ \int \frac{1}{1 + x^2} \, dx = \arctan x \]
\[ \int \frac{1}{\sqrt{1 - x^2}} \, dx = \arcsin x \]
\[ \int \frac{-1}{\sqrt{1 - x^2}} \, dx = \arccos x \]
The one main integration technique we learned in 1210 was integration by substitution. It is also the main topic of section 7.1.

Example:

\[
\int 2x \sin x^2 \, dx = \int \sin u \, du = -\cos u = -\cos x^2 + C
\]

\[
\begin{align*}
    u &= x^2 \\
    \frac{du}{dx} &= 2x \\
    du &= 2x \, dx
\end{align*}
\]

\[
\int f'(g(x)) g'(x) \, dx = f(g(x))
\]

\[
\frac{d}{dx}(-\cos x^2) = -(-\sin x^2) \cdot 2x \\
    = 2x \sin x^2
\]
• For indefinite integrals the substitution is just a tool on the way to the answer which will be in terms of the original integration variable.

For definite integrals we usually do not go back to the original variable, but we must change the limits of integration.

• For example:

\[
\int_0^\sqrt{\pi} 2x \sin x^2 \, dx = \int_0^\pi \sin u \, du
\]

\[u = x^2, \quad 2x \, dx = du\]

\[= -\cos u \bigg|_0^\pi\]

\[= 1 + 1 = 2\]

\[= -\cos x^2 \bigg|_0^\sqrt{\pi}\]
\[ \int \frac{x}{\cos^2 x^2} \, dx = \frac{1}{2} \int \frac{du}{\cos^2 u} = \frac{1}{2} \tan u \]

\[ U = x^2 \]

\[ du = 2x \, dx \]

\[ \frac{1}{2} du = x \, dx \]

\[ \frac{d}{dx} \tan x^2 = \frac{d}{dx} \frac{\sin x}{\cos x} \]

\[ = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \]

\[ = \frac{1}{\cos^2 x} \]

\[ \frac{d}{dx} \frac{1}{2} \tan x^2 = \frac{1}{2} \frac{2x}{\cos^2 x^2} \]

\[ = \frac{x}{\cos^2 x^2} \]
\[
\int \frac{6e^{1/x}}{x^2} \, dx = -\int 6e^u \, du = -6e^u = -6e^{1/x}
\]

\[u = \frac{1}{x}\]

\[\frac{du}{dx} = \frac{d}{dx} x^{-1} = -x^{-2} = -\frac{1}{x^2}\]

\[du = -\frac{dx}{x^2}\]

\[\frac{d}{dx} -6e^{1/x} = -6e^{1/x} \cdot \left(-\frac{1}{x^2}\right) = \frac{6e^{1/x}}{x^2}\]
\[ \int_2^5 t \sqrt{t^2 - 4} dt = \frac{1}{2} \int_0^{21} \sqrt{u} \, du \]

Let \( u = t^2 - 4 \)

\( du = 2t \, dt \)

\[ t \, dt = \frac{1}{2} \, du \]

\[ = \frac{1}{2} \int_0^{21} u^{\frac{1}{2}} \, du \]

\[ = \frac{1}{2} \cdot \frac{2}{3} u^{\frac{3}{2}} \bigg|_0^{21} \]

\[ = \frac{1}{3} \left( 21^{\frac{3}{2}} - 0^{\frac{3}{2}} \right) \]

\[ = \frac{1}{3} \cdot 21^{\frac{3}{2}} \]
\[
\int x^3(x^4 + 1)^{5/2} \, dx = \frac{1}{4} \int u^{5/2} \, du
\]

\[
\begin{align*}
\text{Let } u &= x^4 + 1 \\
\text{Then } du &= 4x^3 \, dx \\
x^3 \, dx &= \frac{1}{4} \, du
\end{align*}
\]

\[
\frac{1}{4} \cdot \frac{2}{2} u^{7/2} = \frac{1}{14} (x^4 + 1)^{7/2}
\]

\[
\frac{d}{dx} \frac{1}{14} (x^4 + 1)^{7/2} = \frac{1}{14} \cdot \frac{7}{2} (x^4 + 1)^{5/2} \cdot 4x^3
\]

\[
= x^3 (x^4 + 1)^{5/2}
\]
\[
\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \int \frac{1}{a \sqrt{1-\left(\frac{x}{a}\right)^2}} \, dx \\
= \int \frac{1}{\sqrt{1-u^2}} \, du \\
\begin{align*}
u &= \frac{x}{a} \\
du &= \frac{1}{a} \, dx \\
= \arcsin \frac{x}{a}
\end{align*}
\]

\[
\frac{d}{dx} \arcsin \frac{x}{a} = \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \\
= \frac{1}{a \sqrt{1-\left(\frac{x}{a}\right)^2}} \\
= \frac{1}{\sqrt{a^2-x^2}}
\]
7.2 Integration by Parts

- A very major topic!
- Based on the Product Rule

\[(fg)' = f'g + fg'\]  \hspace{1cm} (1)

implies

\[fg = \int f'g + fg'\]

- That formula is not very useful. However, subtracting \(fg'\) in (1) gives

\[f'g = (fg)' - fg'\]

which leads to

\[\int f'g = fg - \int fg'\]

and its definite integral version

\[\int_{a}^{b} f'g = fg\big|_{a}^{b} - \int_{a}^{b} fg'\]

- The last two formulas are important and have lots of applications.
Example:
\[
\int x e^x \, dx = \frac{1}{2} x e^x - \int \frac{1}{2} x e^x \, dx
\]

\[
= x e^x - \left( \frac{1}{2} \right) e^x
\]

\[
= x e^x - e^x
\]

\[
= e^x (x - 1)
\]

C B D:
\[
\frac{d}{dx} e^x (x - 1) = e^x (x - 1) + e^x
\]

\[
= e^x - e^x + e^x
\]

\[
= e^x x
\]