6.9 Hyperbolic Functions and their Inverses

- The hyperbolic sine, cosine, and tangent, are defined by

\[
\sinh x = \frac{1}{2} (e^x - e^{-x}) \\
\cosh x = \frac{1}{2} (e^x + e^{-x}) \\
\tanh x = \frac{\sinh x}{\cosh x}
\]

- You might say these are just combinations of exponentials, so they don’t deserve treatment as a separate subject.

- However, they come up a lot in applications, and they have a large number of useful properties. We’ll look at some of them.
Graphs

\begin{figure}
\centering
\includegraphics[width=\textwidth]{hyperbolic_graph.png}
\caption{Graphs of hyperbolic functions.}
\end{figure}
Figure 2. Graphs of hyperbolic functions.
Why “Hyperbolic”? 
Derivatives

\[
\frac{d}{dx} \sinh x =
\]

\[
\frac{d}{dx} \cosh x =
\]

\[
\frac{d}{dx} \tanh x =
\]
Derivatives of Inverses

• cosh is not invertible.
• Let’s do sinh.
• Differentiate implicitly:
• We can compute an explicit expression for \( \sinh^{-1} \):
• and differentiate it explicitly:
\[
\frac{d}{dx} \ln (x + \sqrt{1 + x^2}) =
\]
A Catenary

- A hanging rope suspended at \( x = -a \) and \( x = a \) assumes the shape of the catenary

\[
y(x) = a \cosh \frac{x}{a}
\]

Figure 3. Catenary with \( a = 1 \).
• Find the length of the catenary

\[ y(x) = a \cosh \frac{x}{a}, \quad -a \leq x \leq a. \]

• Expectations?