Salt after 40 min:

$s(t) = \text{amount of salt after } t \text{ minutes}$

$s'(t) = 8 - \frac{5.4}{200}$

$s' = 8 - \frac{s}{50}$
$y + P(x)y = Q(x)$

$S' + \frac{S}{50} = 0.1 \cdot e^{+150} \quad \text{or} \quad e^{+150}$

$e^{+150}S' + \frac{S \cdot e^{+150}}{50} = 8 \cdot e^{+150}$

$S$
Inverse Trig Functions

- Notation:

\[
\begin{align*}
\text{arcsin } x &= \text{asin} x = \sin^{-1} x \\
\text{arccos } x &= \text{acos} x = \cos^{-1} x \\
\text{arctan } x &= \text{atan} x = \tan^{-1} x
\end{align*}
\]

- We won’t bother with the inverses of secant, cosecant, and cotangent.

Trigonometric functions (sin, cos, tan) are not invertible!

- There are infinitely many angles with the same sin, cos, or tan. Those functions fail the horizontal line test!

- To get what is commonly called an inverse trig function we need to restrict the domain of the original trig function!
The arctan function returns an angle in the interval \([-\frac{\pi}{2}, \frac{\pi}{2}]\). If you have an angle outside that interval you cannot possibly get

\[
\arctan \tan x = x
\]

- This is a famous error source, even for graduate students, and perhaps professionals.
Figure 1. \( \tan \) and \( \arctan \).
Figure 2. \text{arcsin, arccos, arctan.}
• OK, but what about derivatives?
• Recall the derivatives of the trig functions:

\[
\frac{d}{dx} \sin x = \cos x \\
\frac{d}{dx} \cos x = -\sin x \\
\frac{d}{dx} \tan x = \frac{1}{\cos^2 x}
\]
\[
\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1 - x^2}}
\]

Aside: \[ (f^{-1})'(f(x)) = \frac{1}{f'(x)} \]

\[
\sin \arcsin x = x
\]

\[
\cos (\arcsin x) \arcsin' x = \frac{d}{dx}
\]

\[
\arcsin' x = \frac{1}{\cos (\arcsin x)}
\]

\[
= \frac{1}{\sqrt{1 - \sin^2 \arcsin x}}
\]

exercise:

\[
\sin^2(\arcsin x) = \left(\sin \arcsin x\right)^2 = x^2
\]

\[
\frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1 - x^2}} = -\frac{d}{dx} \arcsin x
\]
Figure 3. \[ \frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}. \]
Figure 4. $\frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}}$. $\Rightarrow \frac{1}{d x} \arcsin x$
How can arcsin and arccos have derivatives that are negatives of each other? Does that mean

\[
\text{arcsin } x = - \text{arccos } x?
\]

- Obviously not!

\[
\text{arcsin } x + \text{arccos } x = \frac{\pi}{2}
\]

exercise!
\[ \frac{d}{dx} \arctan x = \frac{1}{1 + x^2} \]

\[ \tan \arctan x = x \]

\[ \arctan x = \cos^2(\arctan x) \]
\[
\cos^2 z = \frac{\cos^2 z}{\sin^2 z + \cos^2 z} \quad \frac{1}{\tan^2 z + 1}
\]

\[
\cos^2 \arctan x = \frac{1}{\tan^2 \arctan x + 1} \quad \frac{1}{x^2 + 1} \quad \frac{1}{1 + x^2}
\]
Figure 5. \( \frac{d}{dx} \arctan x = \frac{1}{1+x^2} \).
Examples

\[
\frac{d}{dx} \arcsin x^2 = \frac{2x}{\sqrt{1-x^4}}
\]

\[
\frac{d}{dx} \arcsin(3x + 4) = \frac{3}{\sqrt{1-(3x+4)^2}}
\]

\[
\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin \pi \quad \left( \arcsin \left( \frac{\pi}{2} \right) + \frac{\pi}{2} = \pi \right)
\]

\[
\int \frac{1}{1+x^2} \, dx = \arctan x
\]

\[
\int \frac{1}{4x^2+1} \, dx = \frac{1}{2} \arctan 2x + C
\]

\[
\text{Ex. : } u = 2x
\]
\[ u = 2x \]
\[ du = 2 \, dx \]
\[ dx = \frac{1}{2} \, du \]

\[ I = \frac{1}{2} \int \frac{1}{u^2 + 1} \, du \]
\[ = \frac{1}{2} \arctan u + \zeta_i \]
\[ = \left( \frac{1}{2} \arctan 2x \right) + \zeta_i \]

C B D
A related rates problem: A man standing on top of a vertical cliff is 200 feet above a lake. As he watches, a motor boat moves directly away from the foot of the cliff at a rate of 25 ft/sec. How fast is the angle of depression of his line of sight changing when the boat is 150 feet away from the foot of the cliff?

\[
\frac{h}{s} = \tan \theta \\
\theta = \arctan \frac{h}{s}
\]
\[ \theta' = \frac{-\frac{h}{s^2} s'}{\frac{h^2}{s^2} + 1} \]

\[ = \frac{-h s'}{h^2 + s^2} \]

\[ = \frac{-200 \cdot 25}{200^2 + 150^2} \]

W.T.I
The derivative of the arctan function is rational. (By comparison, the derivatives of arcsin and arccos are radical.)

- That makes the formulas

\[
\frac{d}{dx} \arctan x = \frac{1}{1 + x^2}
\]

and, correspondingly,

\[
\int \frac{1}{1 + x^2} \, dx = \arctan x
\]

important for integration.

- More examples if time allows.

\[
\int \frac{1}{x^2 + 1} \, dx = \arctan x + C
\]

\[
\int \frac{1}{x^2 + 2x + 2} \, dx = \int \frac{1}{(x+1)^2 + 1} \, dx
\]

\[
= \arctan(x+1)
\]
\[
\int \frac{7}{x^2 - 6x + 25} \, dx = \int \frac{7}{x^2 - 6x + 25} \, dx
\]

\[
= \int \frac{7}{x^2 - 6x + 9 + 16} \, dx
\]

\[
= \int \frac{7}{(x-3)^2 + 16} \, dx
\]

\[
= \frac{1}{16} \int \frac{7}{\left(\frac{x-3}{4}\right)^2 + 1} \, dx
\]

\[
= \frac{7}{16} \int \frac{1}{\left(\frac{x-3}{4}\right)^2 + 1} \, dx
\]

\[
= \frac{7}{16} \arctan \left(\frac{x-3}{4}\right) + C
\]

\[
\overline{CBD}
\]
\[ \int \frac{2x+1}{x^2+1} \, dx = \]