Math 1220-3

Notes of 9/6/23

• Remember:

  Study session tomorrow, 9:40, on Zoom!

6.7 Approximations of Differential Equations

DE = Differential Equation

• U of U Math courses on DEs:

  2250  2280  3140  3150  5410  5420  5440
  5470  5500  5620  6410  6420  6430  6440
  6620  6630  6750  6840  6845  6850  6865

• All of those courses are at least half focused on DEs, most are devoted completely to DEs.

• DEs are important because they can be used to model natural processes.

• Ask professors of your major about the importance of DEs.

• Most DEs cannot be solved analytically!

• On the other hand, DEs are used ubiquitously to model natural processes.
• So we need to solve them approximately.
• Numerical solution of DEs is a truly huge field!
• We’ll just look briefly at a few key ideas.
• We consider the DE

\[ y' = f(x, y) \]

• For any point \((x, y)\) there is a function \(y(x)\) that satisfies this equation.
• Example

\[ y' = \frac{1}{5}xy \text{ and } (x, y) = (1, 4) \quad (1) \]

• There is a function \(y = y(x)\) such that

\[ y(1) = 4 \text{ and } y'(1) = \frac{1}{5} \times 1 \times 4 = \frac{4}{5}. \]

• We can reflect this fact by drawing a short line segment of slope \(\frac{4}{5}\) through the point \((1, 4)\).
• In fact, we can do this for points on regular grid and obtain a slope field.
• This is best done by computer.
• Maple, Mathematica, and Matlab are computer codes that can do this and that are available on our computers.
Figure 1. $y' = \frac{xy}{5}$. 
- Figure 1 shows a slope field for the equation P1.
- We can use the slope field to plot approximations of some solutions.

**Figure 2.** Some Solutions of $y' = xy/5$. 

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The backward orbit from (9.7, -0.82) left the computation window.
Ready.
The forward orbit from (9.9, 2.3) left the computation window.
The backward orbit from (9.9, 2.3) left the computation window.
Ready.
• For example, Figure 2 shows some solutions of \( y' = xy/5 \)

• In this particular case we can also solve the DE analytically:

\[
\frac{dy}{dx} = \frac{xy}{5} \quad \left\{ \begin{array}{l}
\int dx = \frac{1}{5} \int x
dx = \frac{x^2}{2} + C \\
\ln y = \frac{x^2}{5} + C \\
y = e^{x^2/5} e^C = e^{x^2/5} \frac{x}{5} = \frac{xy}{5}
\end{array} \right.
\]

\[
x = 1 \quad y = 4 \\
y = e^{\ln 4} = 4 \\
c = \frac{4}{e^{\ln 4}} = 4 \\
c = \ln \frac{4}{e^{\ln 4}} = 0
\]

\[
y = \ln g(x) \\
dy = \frac{c}{dx} \ln g(x) = \frac{g'(x)}{g(x)}
\]

\[
g(x) = y \\
\frac{d}{dx} \ln y = \frac{y'}{y}
\]
\[ \frac{df}{dx} f(g(x)) = f'(g(x))g'(x) \]

\[ \frac{dl}{dx} \ln x = \frac{1}{x} \]

\[ \Rightarrow \frac{dl}{dx} \ln y(x) = \frac{1}{y(x)} \cdot y'(x) \]

\[ m = \frac{y_n - y_{n-1}}{h} = f(x_{n-1}, y_{n-1}) \]

\[ y_n - y_{n-1} = h f(x_{n-1}, y_{n-1}) \]

\[ y_n = y_{n-1} + h f(x_{n-1}, y_{n-1}) \]
The solutions shown in Figure 2 were of course computed by computer, specifically by the code dfield8.m in Matlab.

How would a computer do this?

**Euler’s Method**

Leonard Euler, Swiss Mathematician, 1707-1783, the most prolific mathematician ever, one of the greatest in history.

Basic idea: follow the tangent for a little while.
• **Euler’s Method**

0. Pick a step-size \( h \). Then for \( n = 1, 2, \ldots \)

1. Set \( x_n = x_{n-1} + h \)

2. Set \( y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}) \)

• Example: \( y' = y, \ y(0) = 1, \ h = 0.2 \)

• We get

\[
y_n = y_{n-1} + h y_{n-1} = (1 + h) y_{n-1}
\]

• Clearly

\[
y_n = (1 + h)^n
\]

• See the Table on page 361 of the textbook.

<table>
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<tr>
<th>( n )</th>
<th>( x_n )</th>
<th>( y_n )</th>
<th>( e^{x_n} )</th>
<th>( e^{x_n} - y_n )</th>
</tr>
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<td>1.0</td>
<td>2.48832</td>
<td>2.71828</td>
<td>0.22996</td>
</tr>
</tbody>
</table>

• Let’s focus on the approximation of \( y(1) = e \).
What if we take more steps with smaller step sizes.

• So let

\[
h = \frac{1}{n} \quad \text{and} \quad y_n = (1+h)^n = \left(1 + \frac{1}{n}\right)^n \approx e
\]
• How well does that work?

\[
\begin{array}{ccc}
 n & (1 + \frac{1}{n})^n & e - (1 + \frac{1}{n})^n \\
5 & 2.48832 & 0.22996 \\
10 & 2.59374 & 0.12454 \\
20 & 2.65330 & 0.06498 \\
40 & 2.68506 & 0.03322 \\
80 & 2.70148 & 0.01680 \\
\end{array}
\]

• Every time we double the number of steps and halve the step-size we roughly halve the error.

• We could do better than that. If you are interested take Math 5600 or Math 5620.

• However, we will see later this semester (in section 8.2) that indeed

\[
\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e
\]
- There are very many alternative methods, some of which are hugely sophisticated.
- We’ll just look at some more examples.
- Recall the logistic equation

\[ y' = ky(L - y) \]

\[ y' = ky(1 - y) \quad k = 1 \]

**Figure 3.** \( y' = y(1 - y), \ k = 1, \ L = 1. \)
Figure 3 shows a direction field for $k = 1$ and $L = 1$.

\[ y' = k y (1 - y) \quad k = 1 \]

**Figure 4.** Some solutions of $y' = y(1 - y)$, $k = 1$, $L = 1$.

Figure 4 shows some solutions for $k = 1$ and $L = 1$. 
• If $k$ is increased then the transition from a small population to the limiting population is sharper.

**Figure 5.** $y' = 2y(1 - y), \; k = 2, \; L = 1.$
Figure 5 shows a direction field for $k = 1$ and $L = 1$.

Figure 6. Some solutions of $y' = 2y(1 - y)$, $k = 2$, $L = 1$.

Figure 6 shows some solutions for $k = 2$ and $L = 1$. 
• Compare the Figures 4 and 6 next to each other:

**Figure 7.** $k = 1$.

**Figure 8.** $k = 2$. 
• Consider the differential equation:

\[ y' = \cos(x) + \lambda(y - \sin x). \]

• The general solution of this equation is

\[ y(x) = \sin x + Ce^{\lambda x} \]

• Let’s look at this for some values of \( \lambda \).
Figure 9. $\lambda = 0$. 

The backward orbit from (0.0004, 1)
Ready.
The forward orbit from (0.017, 1.5)
The backward orbit from (0.017, 1.5)
Ready.
Figure 10. $\lambda = 0.5$. 

The backward orbit from $(0.76, -0.74)$
Ready.
The forward orbit from $(0.8, -0.61)$ left the computation window.
The backward orbit from $(0.8, -0.61)$
Ready.
\[ y' = \cos(x) + L (y - \sin(x)) \]

\[ L = -0.5 \]

**Figure 11.** \( \lambda = -0.5 \).
The backward orbit from \((0.12, -0.73)\) left the computation window.
Ready.
The forward orbit from \((0.16, -1.2)\)
The backward orbit from \((0.16, -1.2)\) left the computation window.
Ready.

**Figure 12.** \(\lambda = -0.5\).