Math 1210-23Notes of 1/22–23/24

Announcements

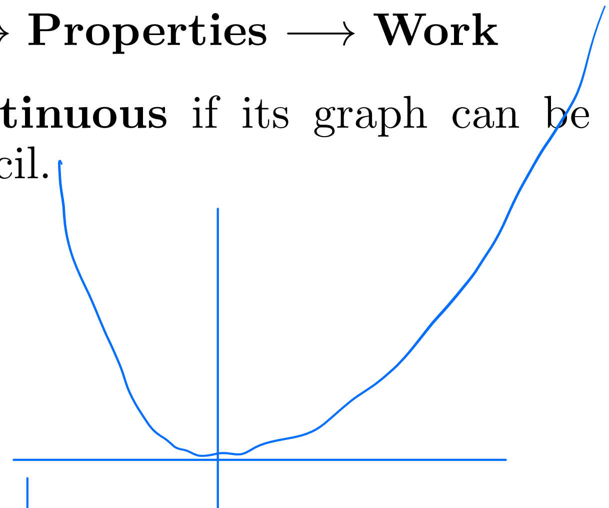
- These notes are for two days. Let's see how far we get today.
- hw 1 closes tonight
- hw 2 closes Wednesday
- New Exam schedule:
  - Exam 1: Friday 2/2/24
  - Exam 2: Friday 3/1/24
  - Exam 3: Friday 4/5/24
  - Final Exam: Monday, April 29, 10:30am-12:30pm, same as before.
- All Exams take place in JTB 310

1.6 Continuity

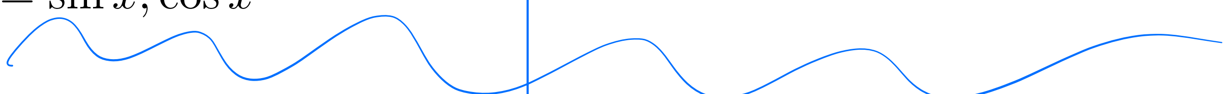
- Last section of chapter 1.
- Remember

  Concept $\longrightarrow$ **Definition** $\longrightarrow$ **Properties** $\longrightarrow$ **Work**
- Intuition: A function is **continuous** if its graph can be drawn without lifting the pencil.
- Continuous or not:

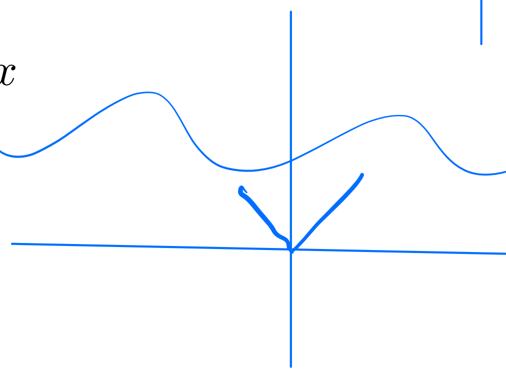
$f(x) = x^2$



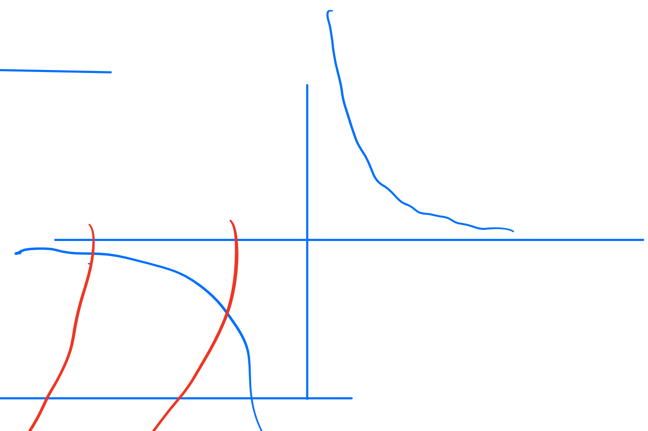
$f(x) = \sin x, \cos x$



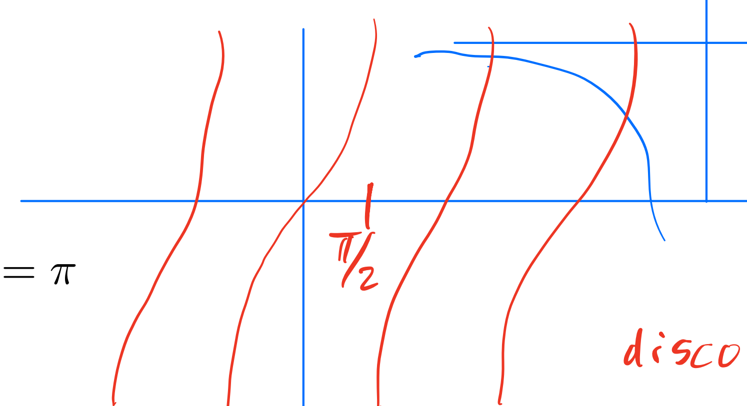
$f(x) = |x|$



$f(x) = \frac{1}{x}$



$f(x) = \tan x$



discontinuous at $x = \frac{\pi}{2} + n\pi$

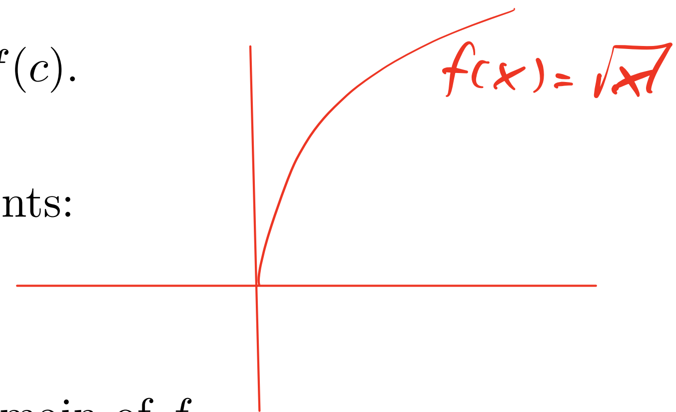
n integer

$$f(x) = \begin{cases} 0 & \text{if } x = \pi \\ 1 & \text{else} \end{cases}$$

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

Definition: We say that the function f is **continuous** at $x = c$ if

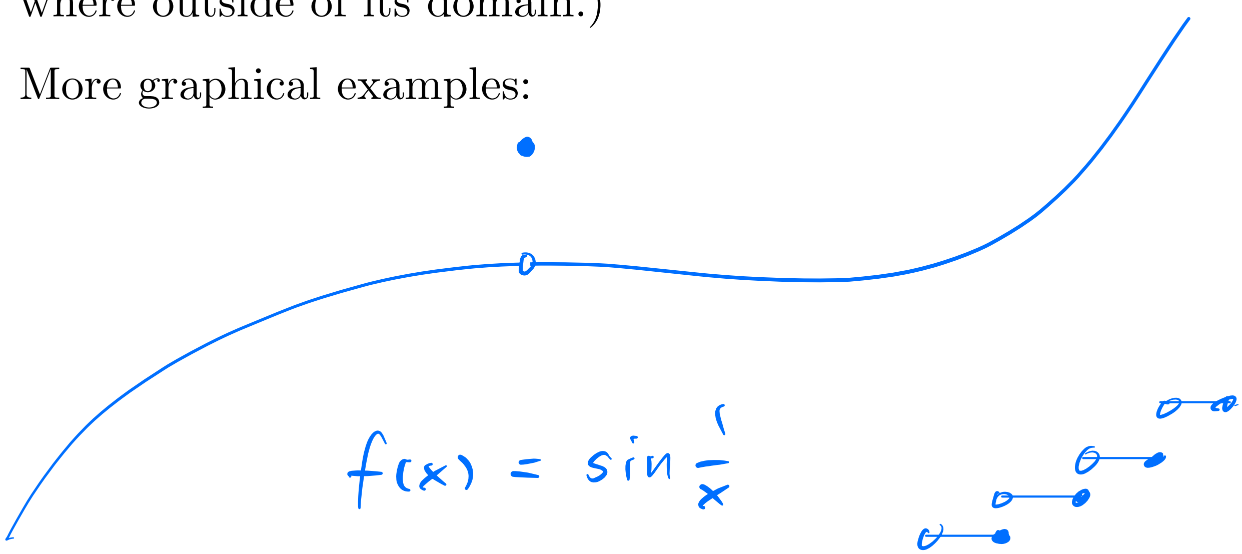
$$\lim_{x \longrightarrow c} f(x) = f(c).$$



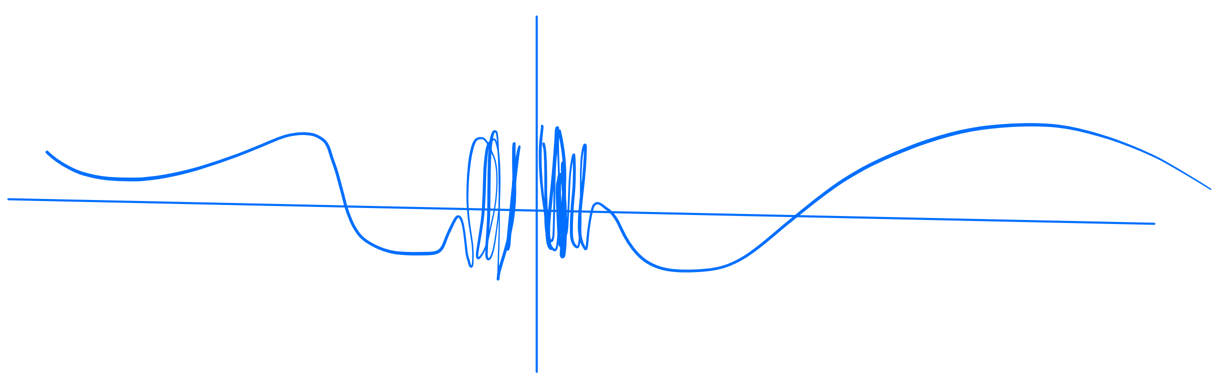
This definition has three ingredients:

1. $\lim_{x \longrightarrow c} f(x)$ exists
2. $f(c)$ is defined, i.e., c is in the domain of f.
3. The equality $\lim_{x \longrightarrow c} f(x) = f(c)$ holds.

- If any of these three items is missing then we say that the function is **discontinuous** at c. (This has the somewhat bizarre consequence that a function is discontinuous everywhere outside of its domain.)
- More graphical examples:



$f(x) = \sin \frac{1}{x}$



- Example 2: How do we define $f(2)$ such that

$$f(x) = \frac{x^2-4}{x-2}$$

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2}$$

$$= \lim_{x\to 2} x+2$$

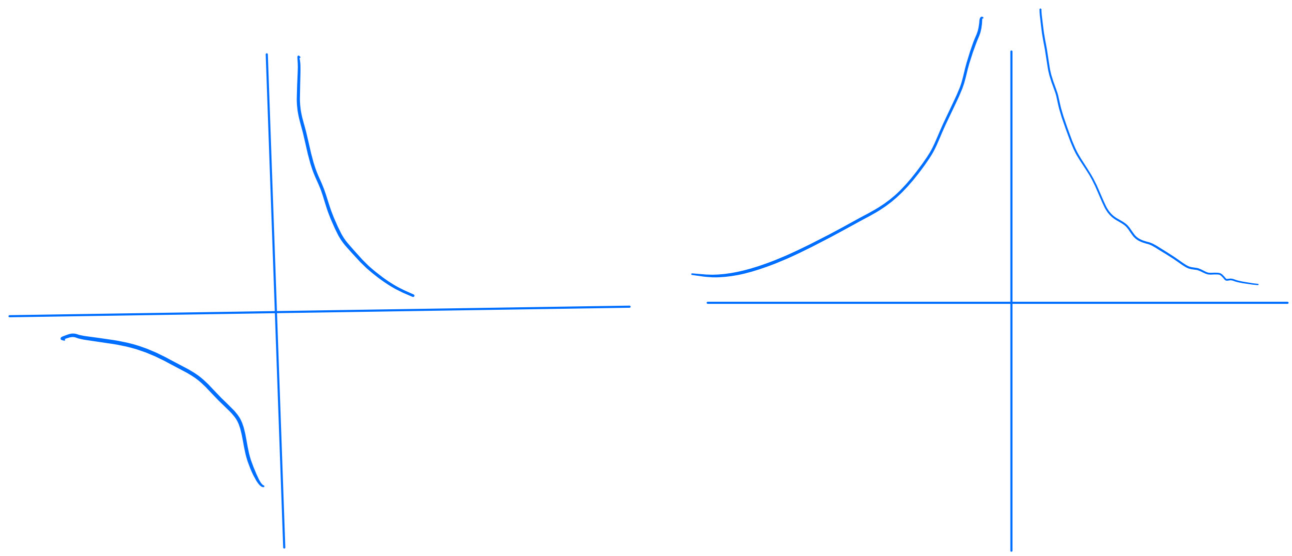
is continuous at $x = 2$.

$$g(x) = \begin{cases} \frac{x^2-4}{x-2} & x \neq 2 \\ 4 & x = 2 \end{cases}$$

$$g(x) = x+2$$

- If we define $f(2)$ to be any value other than 4 the resulting function is discontinuous at $x = 2$. We say that discontinuity is **removable** because it can be removed by changing the value of f at just one value of x. Such a discontinuity is also called a **hole**.

- Example: $f(x) = \frac{1}{x}$ or $f(x) = \frac{1}{x^2}$. These expressions cannot be evaluated when $x = 0$. But no matter how we define $f(0)$, the discontinuity cannot be removed.

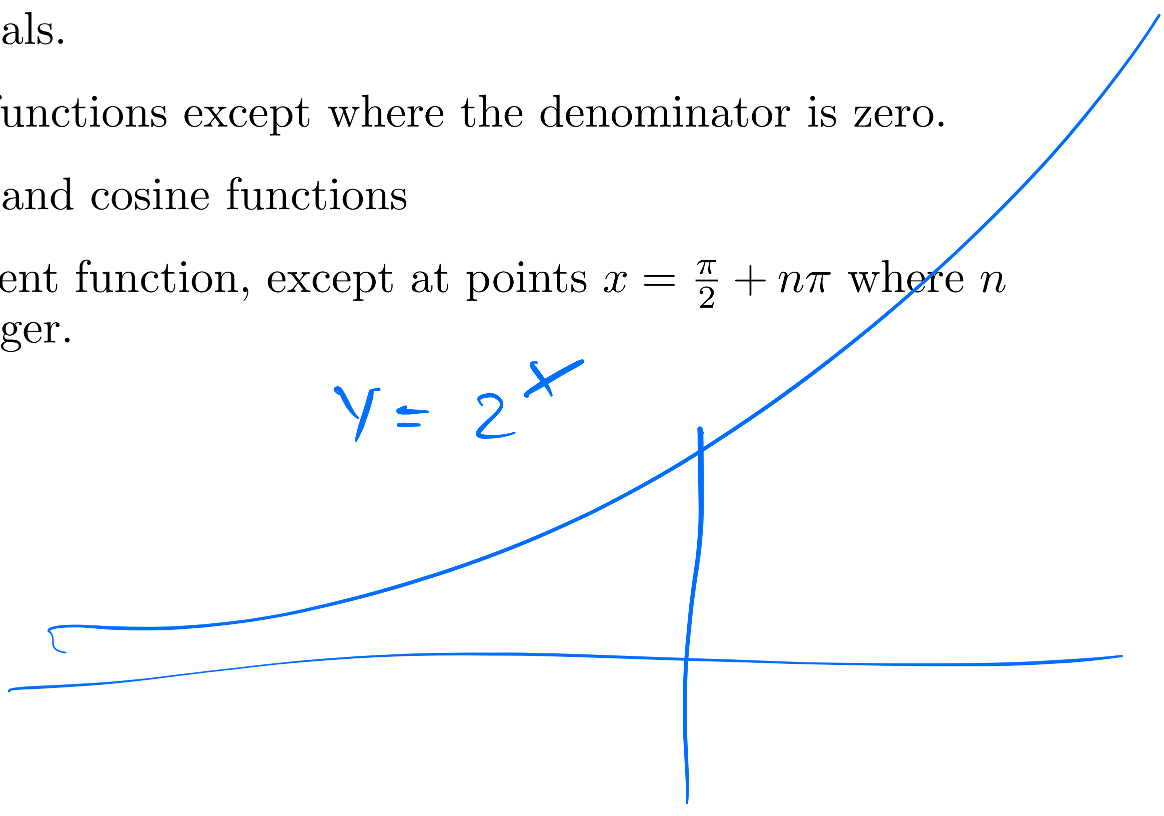


- A function that is continuous at every point in its domain is called **continuous** or, sometimes, **everywhere continuous**.
- Most functions that we will encounter are continuous everywhere, or continuous everywhere with the exception of a few points.
- Continuity of a function can be established by applying the definition, which in turn requires the definition of limits.
- The details are tedious and non-illuminating.
- We provide instead a

Catalog of Continuous Functions

The following functions are continuous for all real numbers with the stated exceptions:

- polynomials.
- rational functions except where the denominator is zero.
- The sine and cosine functions
- The tangent function, except at points $x = \frac{\pi}{2} + n\pi$ where n is an integer.



Combining Functions

$f(x) = x + \frac{1}{1+x^2}$

- Suppose k is a constant and f and g are continuous at a point c. Then so are

$$kf, \qquad f+g, \qquad f-g, \quad fg, \quad \text{and} \quad \frac{f}{g} \quad \text{provided} \quad g(c) \neq 0.$$

- **Major fact:** The composition of continuous functions is continuous. Suppose g is continuous at $x = c$ and f is continuous at $g(c)$. Then

$$\lim_{x \longrightarrow c} (f \circ g)(x) = \lim_{x \longrightarrow c} f\big(g(x)\big) = \underline{f\big(g(c)\big)}.$$

- There is an $\epsilon - \delta$ proof on page 85 of the textbook. We will skip it.

$f \circ g \qquad g \circ f$

$f(x) = \sin x \qquad g(x) = x^2$

$f(g(x)) = \sin(x^2) = \sin x^2$

$g(f(x)) = (\sin x)^2 = \sin^2 x$

One-sided Continuity

- Works the same way, except we replace the limit by a one-sided limit.
- Example. Recall greatest integer function:

  $[\![x]\!]$ = the greatest integer $\leq x$.
- Is the greatest integer function, continuous, discontinuous, left continuous, or right continuous at integers x?

$$\lim_{x \to c} f(x) = f(c)$$

c^+

$[\![x]\!]$ = greatest integer $\leq x$

$$[\![2.1]\!] = 2 = \lim_{x \to 2^+} [\![x]\!] = [\![2]\!]$$

$$[\![1.9]\!] = 1$$

- Exercise: modify the definition of $[\![x]\!]$ so that it is left-continuous rather than right-continuous.

- There are functions that are everywhere discontinuous or nowhere continuous:
- Example:

$$f(x) = \begin{cases} 0 & x \text{ rational} \\ 1 & x \text{ irrational} \end{cases}$$

The Intermediate Value Theorem

- Page 87, textbook: Assume f is continuous

  Let f be defined on $[a, b]$. Suppose

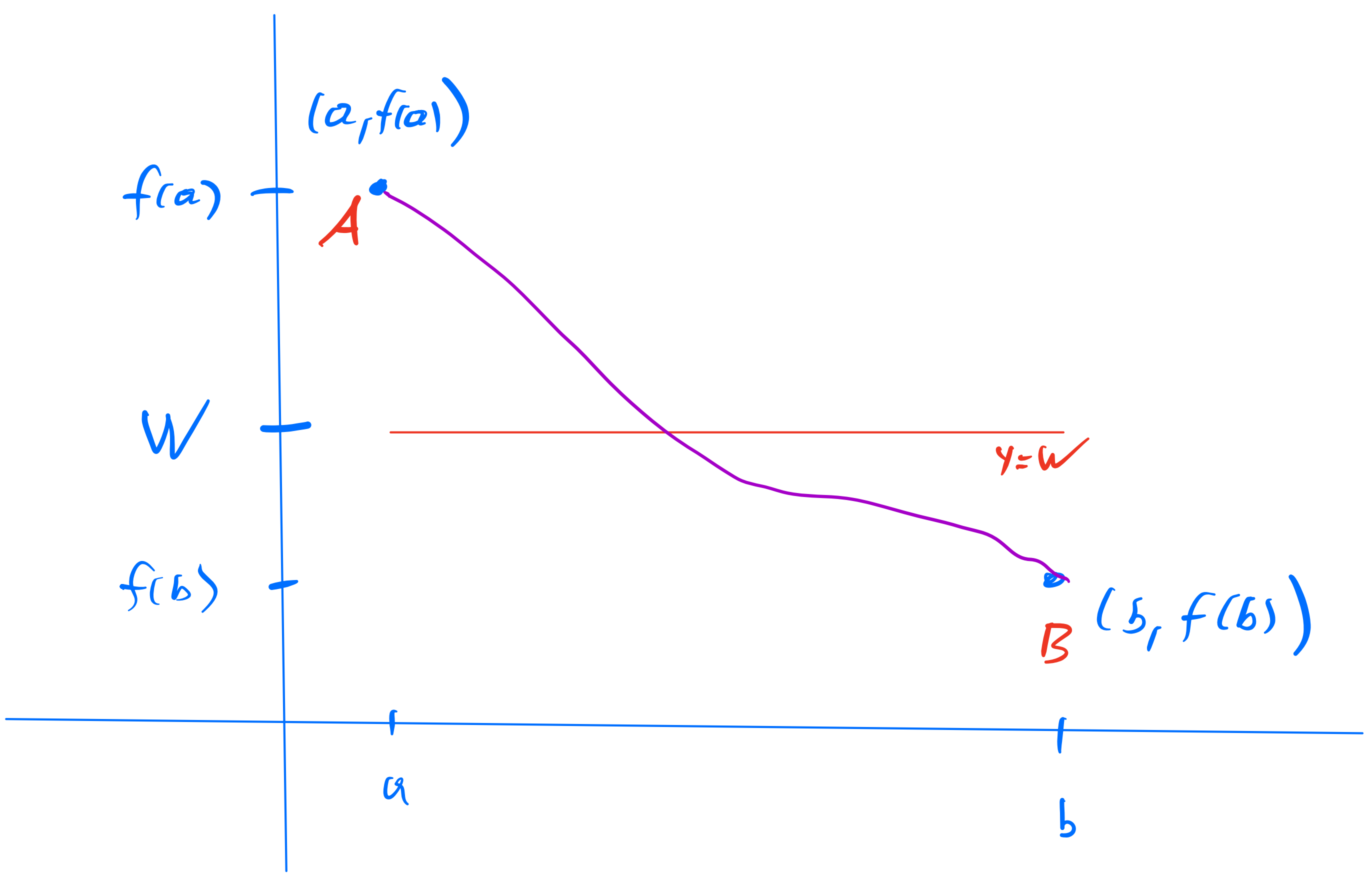
$$f(a) \le W \le f(b).$$

$$\ge \quad \ge$$

Then there is at least one number c such that

$$a \le c \le b \quad \text{and} \quad f(c) = W$$

- This is actually a deep result having to do with the completeness of the real number system. But here is a geometric plausibility argument:



- Example for an application: Show that the equation

$$f(x) = 2^x + x = 0$$

has a solution in the interval $[-1, 0]$. $f(c) = 0$

$f(0) = 2^0 + 0 = 1$ $\quad -\frac{1}{2} < 0 < 1$

$f(-1) = \frac{1}{2} - 1 = -\frac{1}{2}$

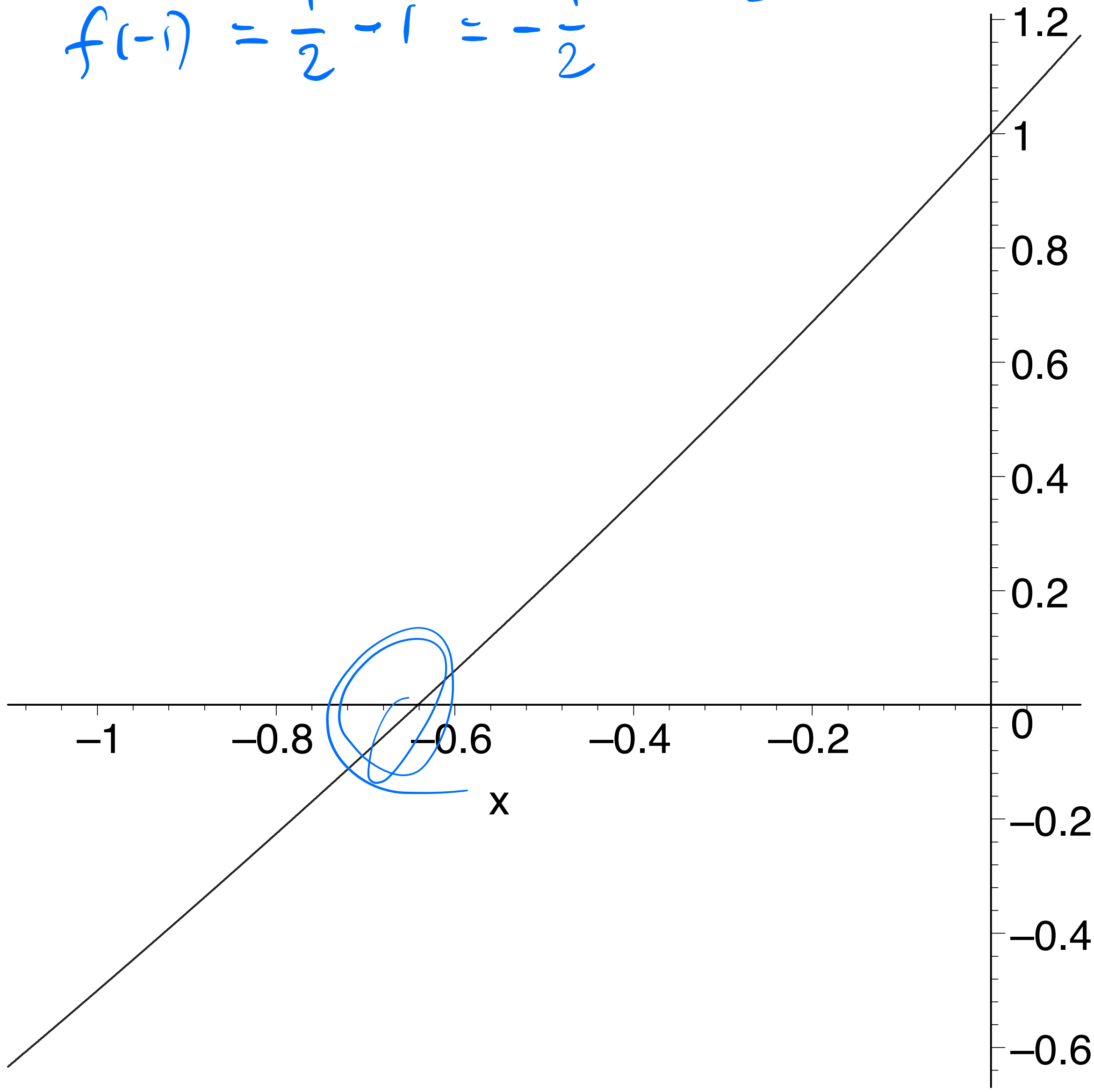


Figure 1. $f(x) = 2^x + x$.

- Another Example: Suppose the temperature on earth is continuous. Can you always find two antipodal points that have the same temperature?

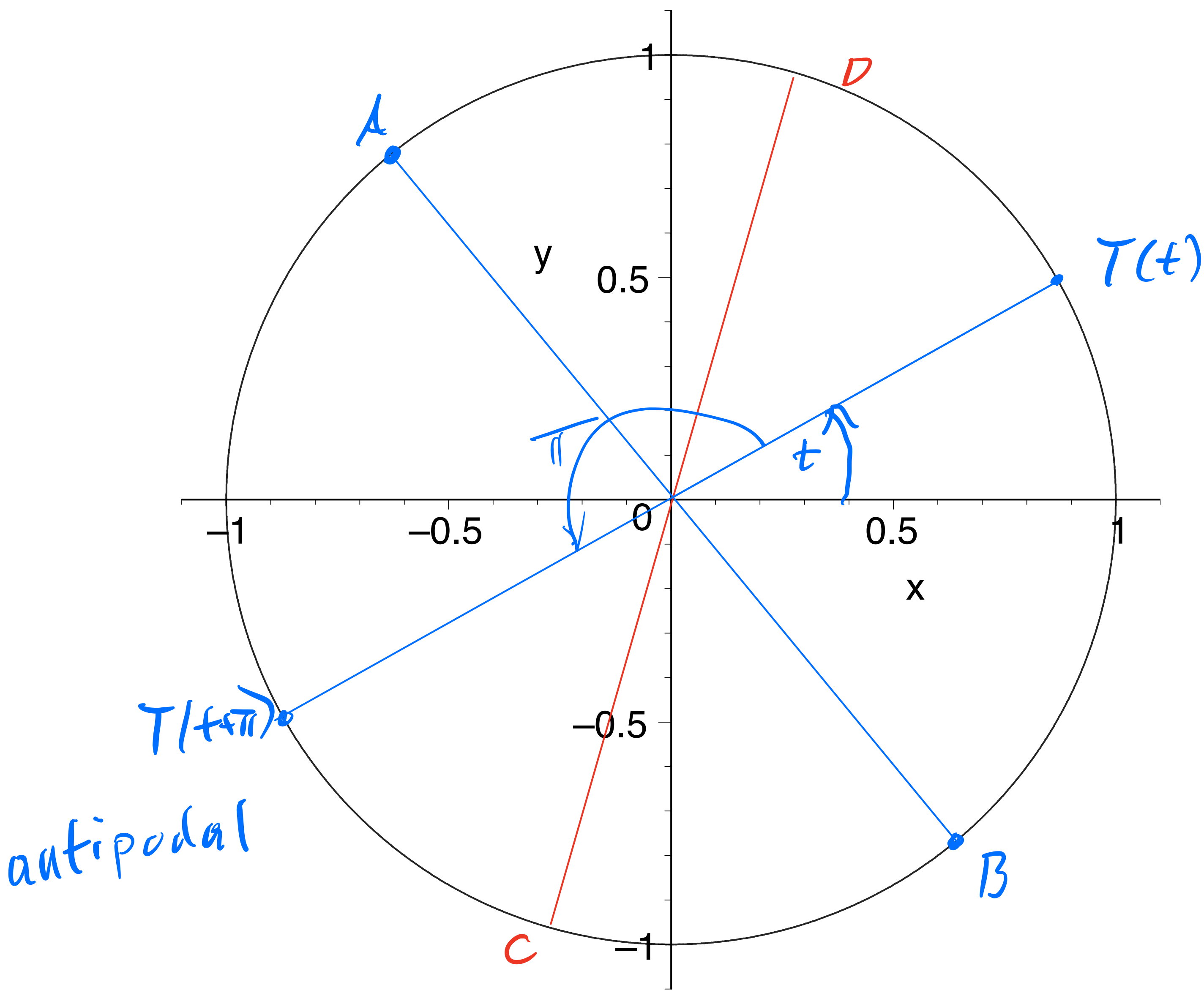


Figure 2. A Great Circle.

$\propto$ $f(t) = T(t) - T(t+\pi)$

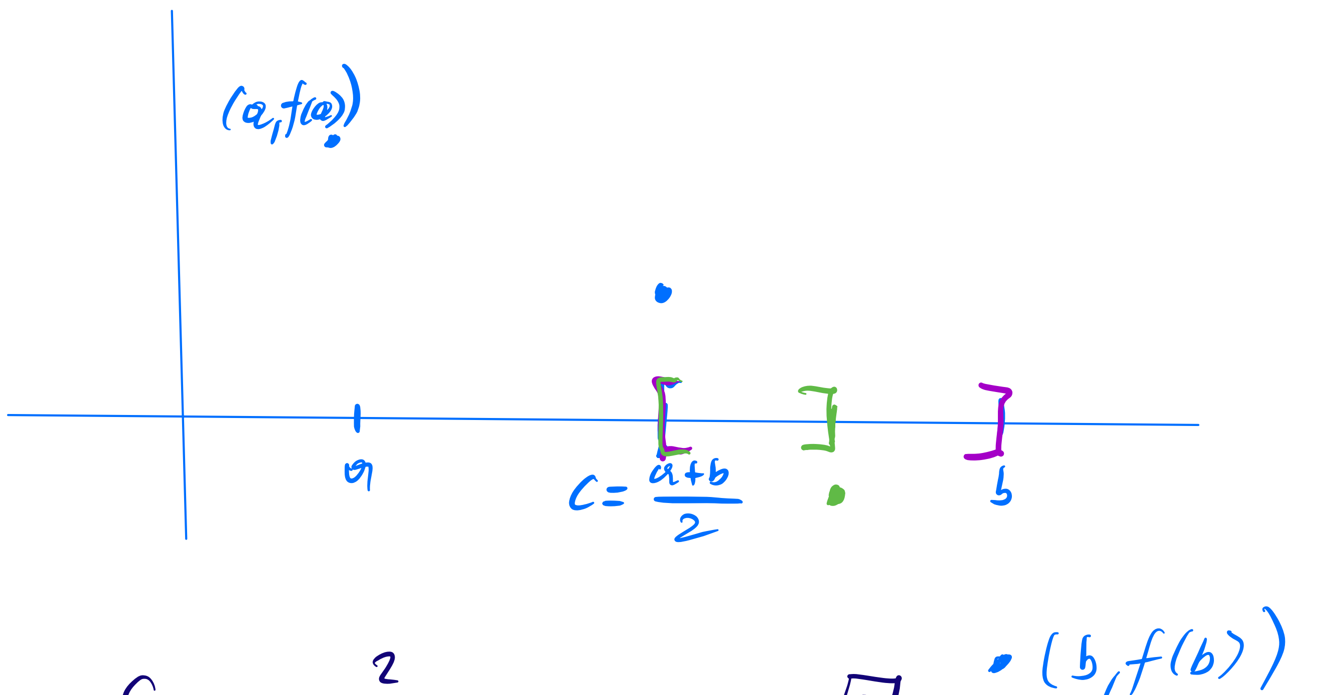
$f(t+\pi) = T(t+\pi) - T(t+2\pi) = T(t+\pi) - T(t) = -f(t) < 0$

The Method of Bisection

f continuous

$f(x) = 0 \qquad x = ?$

$f(a) > 0 \qquad f(b) < 0$



Ex.: $f(x) = x^2 - 2 = 0 \qquad x = \sqrt{2}$

$a = 1 \quad f(1) = -1 \qquad [1, 2]$

$b = 2 \quad f(2) = 2$

$c = \frac{1+2}{2} = \frac{3}{2} \quad f(\frac{3}{2}) = (\frac{3}{2})^2 - 2 = \frac{9}{4} - 2 = \frac{1}{4} > 0 \quad [1, \frac{3}{2}]$

$d = \frac{1}{2}(1 + \frac{3}{2}) = \frac{5}{4} \quad \underline{f(\frac{5}{4})} = (\frac{5}{4})^2 - 2 = \frac{25}{16} - 2 = -\frac{7}{16}$

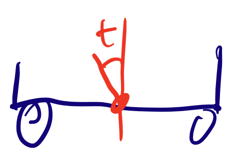
$[\frac{5}{4}, \frac{3}{2}]$

$\div$

The Train Puzzle

- Courant and Robbins, *What is Mathematics*, ISBN 978-0-19-5120519-3, page 319. (Highly recommended, this is the greatest book on the planet!)
- I'll quote from Courant and Robbins:

Suppose a train travels from station A to station B along a straight section of track. The journey need not be of uniform speed or acceleration. The train may act in any manner, speeding up, slowing down, or even backing up for a while, before reaching B. But the exact motion of the train is supposed to be known in advance; that is, the function $s = f(t)$ is given, where s is the distance of the train from station A, and t is the time, measured from the instant of departure. On the floor of one of the cars a rod is pivoted so that it may move without friction either forward or backward until it touches the floor. Is it possible to place the rod in such a position that, if it is released at the instance when the train starts and allowed to move solely under the influence of gravity and the motion of the train, it will not fall to the floor during the entire journey from A to B [and end up being vertical when the train reaches B]?



$f(-\frac{\pi}{2}) = -\frac{\pi}{2}$ $\quad$ $f(\frac{\pi}{2}) = \frac{\pi}{2} = -f(-\frac{\pi}{2})$