Math 1210-23 Notes of 1/19/24

Announcements

- hw 1 due Monday 1/22
- hw 2 due Wednesday 1/24
- hw 3 open, due ~~next Wednesday~~ 1/31

Exam 1 on Chapter 1 ~~next Wednesday~~ 1/31. Finish hw 3 before the exam, even though it only closes that evening!

- Study Session today after class, right here

1.4 Limits Involving Trig Functions

- Recall the definition of angles, sine, and cosine

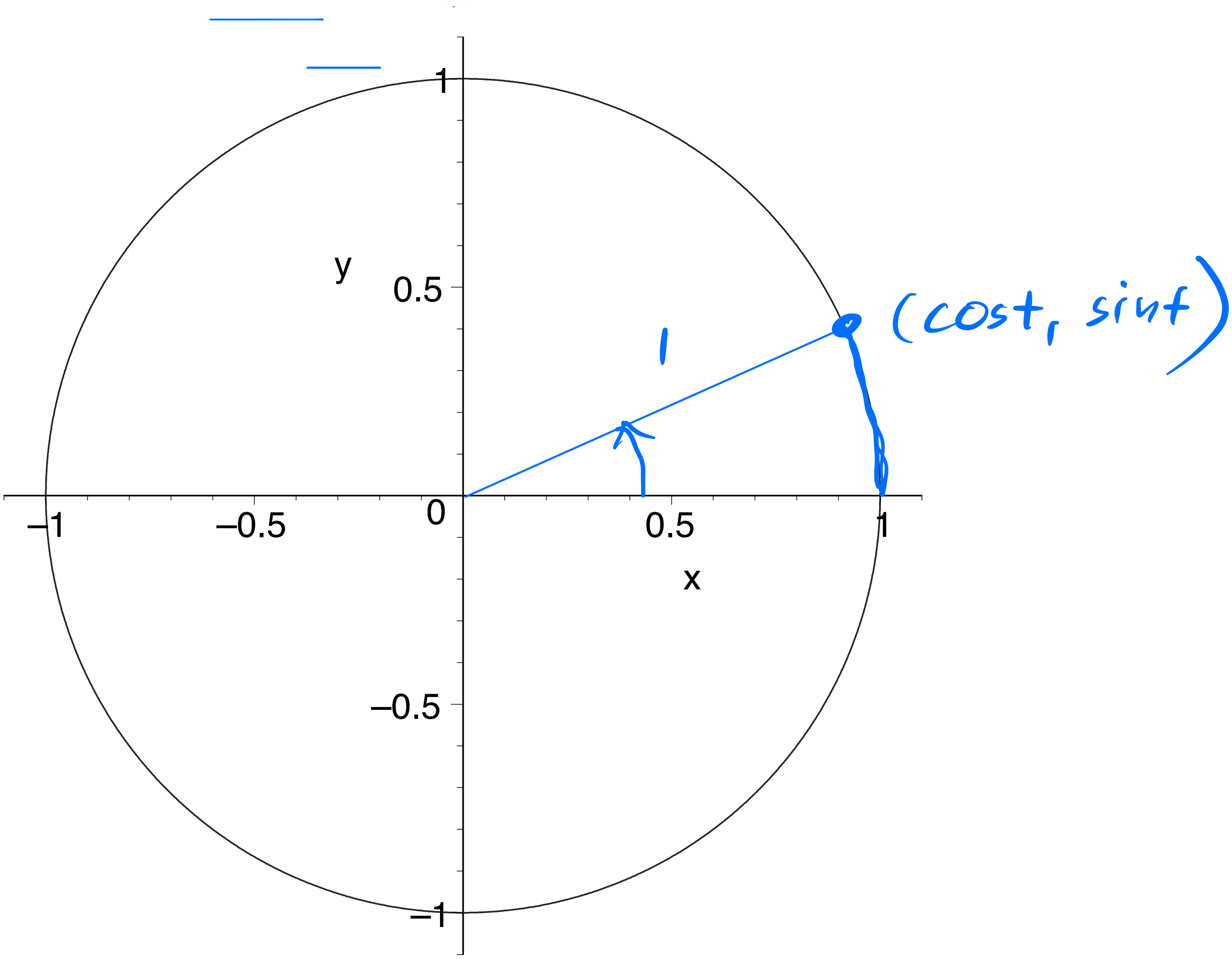


Figure 1. Sine and Cosine.

- Recall

$$\tan t = \frac{\sin t}{\cos t}$$

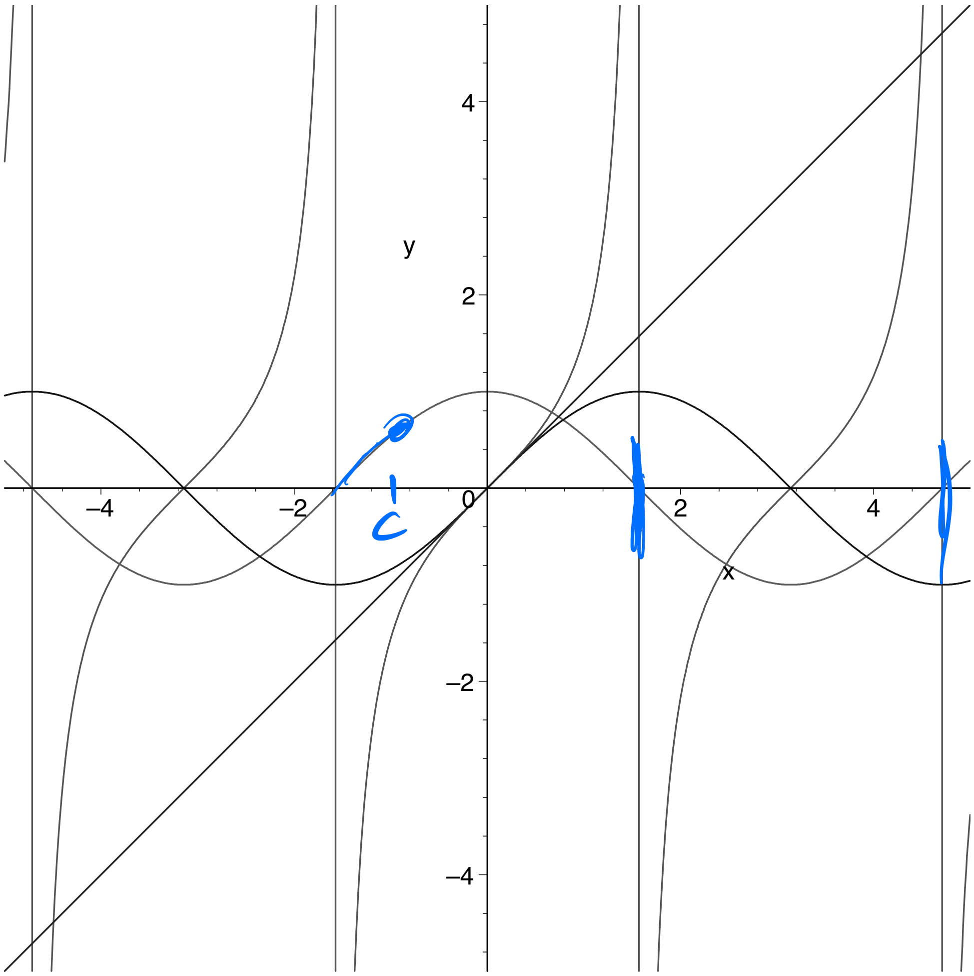


Figure 2. Graphs of sin, cos, tan, x.

- It's clear from the graph that

$$\lim_{t \longrightarrow c} \sin t = \sin c \quad \text{and} \quad \lim_{t \longrightarrow c} \cos t = \cos c$$

- There is a more rigorous discussion (Theorem A, page 74) in the textbook.

- We will need the following limits:

$$\lim_{t\longrightarrow 0}\frac{\sin t}{t}=1 \quad \text{and} \quad \lim_{t\longrightarrow 0}\frac{1-\cos t}{t}=0.$$

- This is the contents of Theorem B on page 75.
- We already saw the first statement:

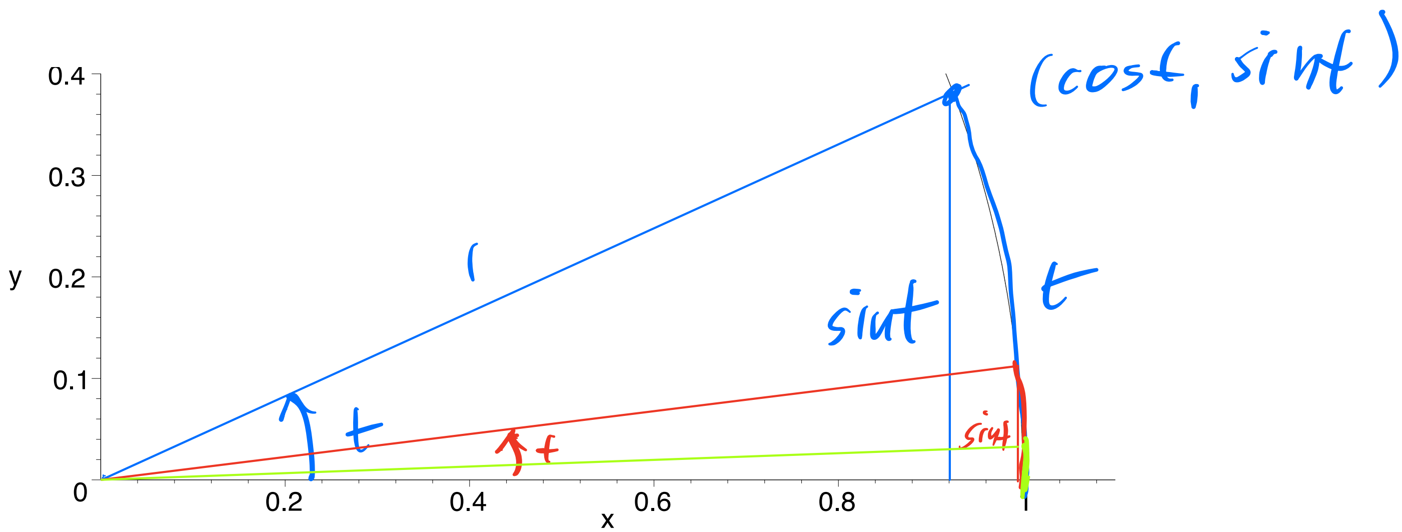


Figure 3. $\lim_{t\longrightarrow 0}\frac{\sin t}{t}=1.$

- Let's look a little closer. Recall

The Squeeze Theorem. Suppose f, g, and h are functions such that

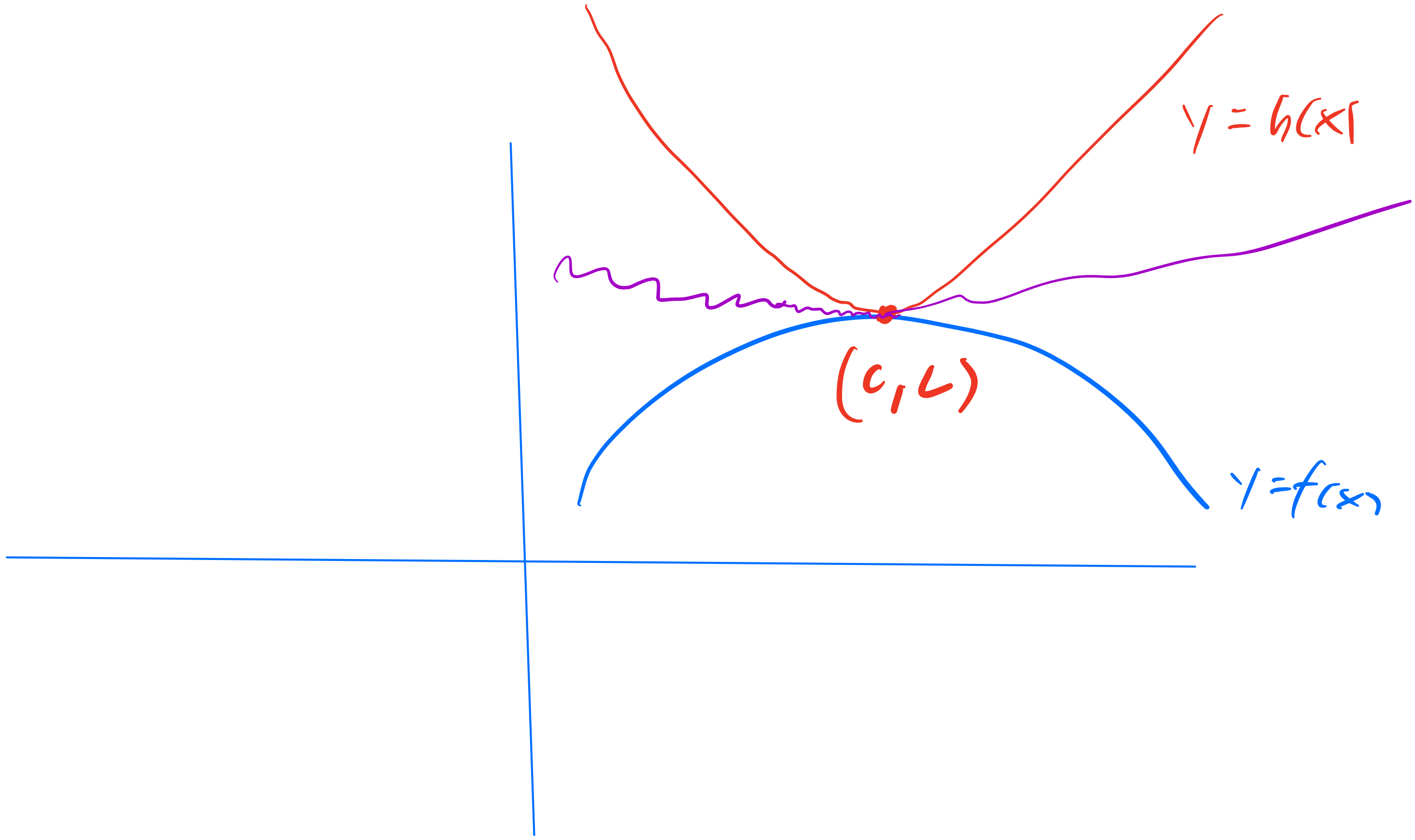
$$f(x)\le g(x)\le h(x)$$

for all x near c except possibly *at* $x=c$. Also assume that

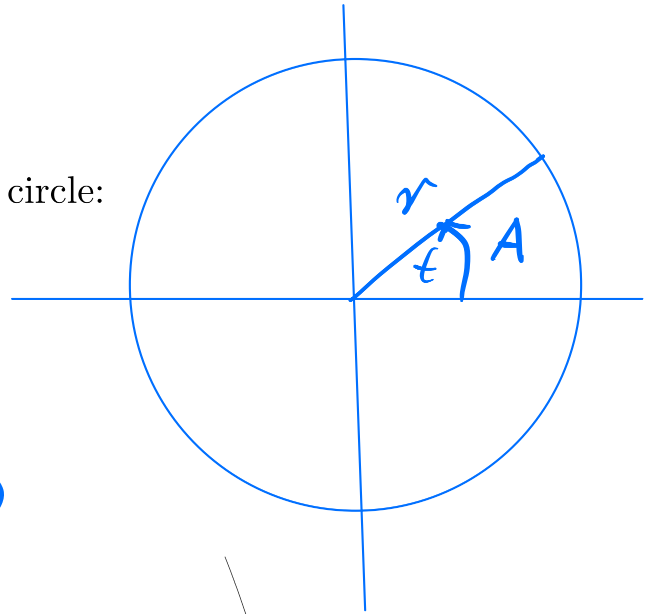
$$\lim_{x\longrightarrow c}f(x)=\lim_{x\longrightarrow c}h(x)=L$$

Then

$$\lim g(x)=L$$


y = h(x)
(c, L)
y = f(x)

- Now recall the area of a sector of a circle:



$$A = \text{area} = \frac{t}{2} r^2.$$

- Apply to the sector shown here:

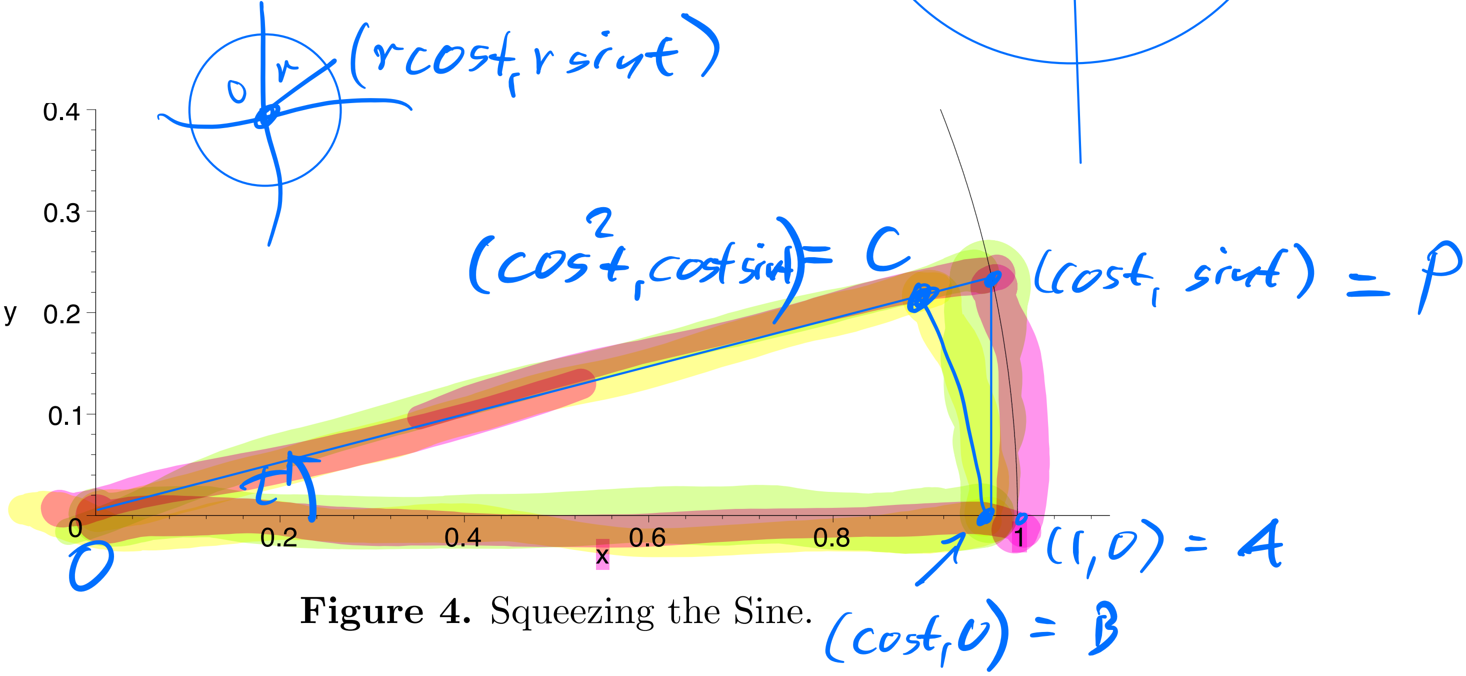


Figure 4. Squeezing the Sine.

$$O = (0,0), \quad A = (1,0), \quad B = (\cos t, 0), \quad P = (\cos t, \sin t), \quad C = (\cos^2 t, \cos t \sin t)$$

and

$$f(t) = \text{area}(\text{sector}OBC) = \frac{t}{2}\cos^2 t$$

$$g(t) = \text{area}(\Delta OBP) = \frac{1}{2}\cos t \sin t$$

$$h(t) = \text{area}(\text{sector}OAP) = \frac{t}{2}.$$

- Clearly

$$f(t) \le g(t) \le h(t)$$

or

$$\frac{t}{2}\cos^2 t \le \frac{1}{2}\cos t \sin t \le \frac{t}{2}.$$

$$\frac{t\cos^2 t \le \cos t \sin t \le t}{t\cos t}$$

Multiply with 2 and divide by $t\cos t$ to get

$$\cos t \le \frac{\sin t}{t} \le \frac{1}{\cos t}$$

Clearly

$$\lim_{t\longrightarrow 0} \cos t = \lim_{t\longrightarrow 0} \frac{1}{\cos t} = 1$$

and, by the Squeeze Theorem

$$\lim_{t\longrightarrow 0} \frac{\sin t}{t} = 1.$$

$(a-b)(a+b)$
$= a^2 - b^2$

- This implies that

$$L = \lim_{t\longrightarrow 0} \frac{1-\cos t}{t} = 0:$$

$$\frac{1-\cos t}{t} = \frac{(1-\cos t)(1+\cos t)}{t(1+\cos t)}$$

$$= \frac{1-\cos^2 t}{t(1+\cos t)}$$

$$= \frac{\sin^2 t}{t(1+\cos t)}$$

$$L = \lim_{t\to 0} \frac{\sin^2 t}{t(1+\cos t)} = \underbrace{\lim_{t\to 0} \frac{\sin t}{t}}_{1} \cdot \underbrace{\lim_{t\to 0} \frac{\sin t}{\cos t + 1}}_{0}$$

$$= 1 \cdot 0 = 0$$

More Limits

$$\lim_{\theta\longrightarrow 0}\frac{\tan\theta}{\theta} = \lim_{\theta \to 0} \frac{\sin\theta}{\theta\cdot\cos\theta} = \underbrace{\lim_{\theta\to 0}\frac{1}{\cos\theta}}_{1}\cdot\underbrace{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}}_{1}$$

$$= 1$$

$$\lim_{x\longrightarrow 0}\frac{\sin x^\circ}{x} = \lim_{x\to 0}\frac{\sin\frac{x\cdot\pi}{180}\ (\text{rad})}{x}$$

$$= \lim_{x\to 0}\frac{\sin\frac{\pi x}{180}}{\frac{\pi x}{180}}\ \frac{\pi}{180} \qquad z = \frac{\pi x}{180}$$

$$= \left(\lim_{z\to 0}\frac{\sin z}{z}\right)\cdot\frac{\pi}{180} = \frac{\pi}{180}$$

$$\lim_{\theta\longrightarrow 0}\frac{\tan 5\theta}{\sin 2\theta} = \lim_{\theta\to 0}\frac{\tan(5\theta)\cdot 2\theta}{5\theta\cdot\sin(2\theta)}\cdot\frac{5}{2}$$

$$= \frac{5}{2}\cdot\underbrace{\lim_{\theta\to 0}\frac{\tan 5\theta}{5\theta}}_{1}\cdot\underbrace{\lim_{\theta\to 0}\frac{2\theta}{\sin 2\theta}}_{1}$$

$$= \frac{5}{2}$$

1.6 Continuity

- Remember

  Concept $\longrightarrow$ Definition $\longrightarrow$ Properties $\longrightarrow$ Work

- Intuition: A function is **continuous** if its graph can be drawn without lifting the pencil.

- Continuous or not:

$f(x) = x^2$

$f(x) = \sin x, \cos x$

$f(x) = |x|$

$f(x) = \frac{1}{x}$

$f(x) = \tan x$

$$f(x) = \begin{cases} 0 & \text{if } x = \pi \\ 1 & \text{else} \end{cases}$$

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$