Announcements

- Schedule of events now on Canvas. Link on Home Page.

1.3, 1.5, Working with Limits

- Recall Procedure:

  Concept $\longrightarrow$ **Definition** $\longrightarrow$ **Properties** $\longrightarrow$ **Work**

- Our **Definition**: We say that the limit of $f(x)$ as x approaches c equals L, or

$$\lim_{x \longrightarrow c} f(x) = L$$

  if for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$0 < |x - c| < \delta \qquad \Longrightarrow \qquad |f(x) - L| < \epsilon.$$

- ϵ is the is the lower case Greek letter *epsilon*, and δ is the lower case Greek letter *delta*.

Properties:

Main Limit Theorem

- See textbook, page 68.
- Let n be a positive integer, k a constant, and f and g functions that have limits at c. Then:

1. $\lim_{x\longrightarrow c} k = k$.

2. $\lim_{x\longrightarrow c} x = c$.

3. $\lim_{x\longrightarrow c} kf(x) = k \lim_{x\longrightarrow c} f(x)$.

4. $\lim_{x\longrightarrow c} \big(f(x) + g(x)\big) = \lim_{x\longrightarrow c} f(x) + \lim_{x\longrightarrow c} g(x)$.

5. $\lim_{x\longrightarrow c} \big(f(x) - g(x)\big) = \lim_{x\longrightarrow c} f(x) - \lim_{x\longrightarrow c} g(x)$.

6. $\lim_{x\longrightarrow c} \big(f(x) \cdot g(x)\big) = \Big(\lim_{x\longrightarrow c} f(x)\Big) \cdot \Big(\lim_{x\longrightarrow c} g(x)\Big)$.

7. $\lim_{x\longrightarrow c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\longrightarrow c} f(x)}{\lim_{x\longrightarrow c} g(x)}$ provided $\lim_{x\longrightarrow c} g(x) \neq 0$.

8. $\lim_{x\longrightarrow c} \big(f(x)\big)^n = \Big(\lim_{x\longrightarrow c} f(x)\Big)^n$.

9. $\lim_{x\longrightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\longrightarrow c} f(x)}$ provided $f(x) \geq 0$ when n is even.

- Consequence of Main Limit Theorem:

$$\lim_{x \longrightarrow c} f(x) = f(\underset{c}{\cancel{x}})$$

if f is a polynomial or a rational function with a non-zero denominator at $x = c$.

- **Example**

$$\lim_{x \longrightarrow 1} \frac{x+4}{x^2+1} = \frac{1+4}{1+1} = \frac{5}{2}$$

$\cancel{2\frac{1}{2}}$ $\cancel{2.5}$

- subtle point, and frequent source of errors: when combining two functions the limit may exist even if the individual limits do not.
- simple example:

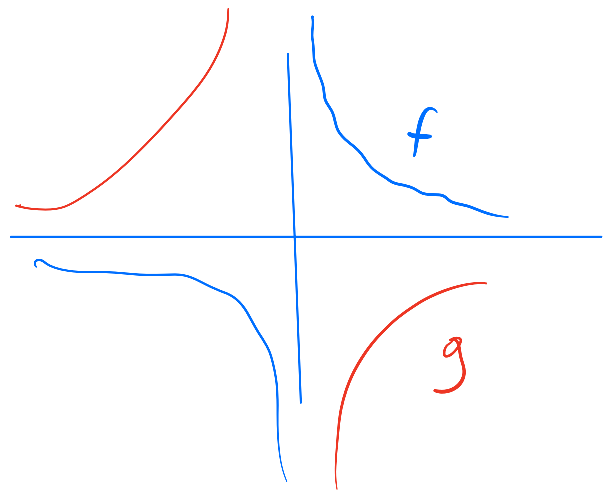
$$f(x) = \frac{1}{x} \quad \text{and} \quad g(x) = -\frac{1}{x}$$

What happens to $f(x)+g(x)$ as x goes to zero? The individual limits do not exist, but the limit of the sum is zero.

$$\lim_{x \to c} f(x) + \lim_{x \to c} g(x) = \lim_{x \to c} (f(x)+g(x))$$

$$\lim_{x \to c} f(x) = \lim_{x \to c} (f(x)+g(x)) - \lim_{x \to c} g(x)$$

$$f(x) = \frac{1}{x}$$



$$h(x) = f(x)+g(x) = \frac{1}{x} + \left(-\frac{1}{x}\right) = 0 \qquad x \neq 0$$

$$\lim_{x \to 0} (f(x)+g(x)) = 0$$

$h(x) =$

- We proved item 4 of the main limit theorem in class.
- Proof of the other parts is in the textbook in section 1.3.
- Another major fact is

The Squeeze Theorem. Suppose f, g, and h are functions such that

$$f(x) \leq g(x) \leq h(x)$$

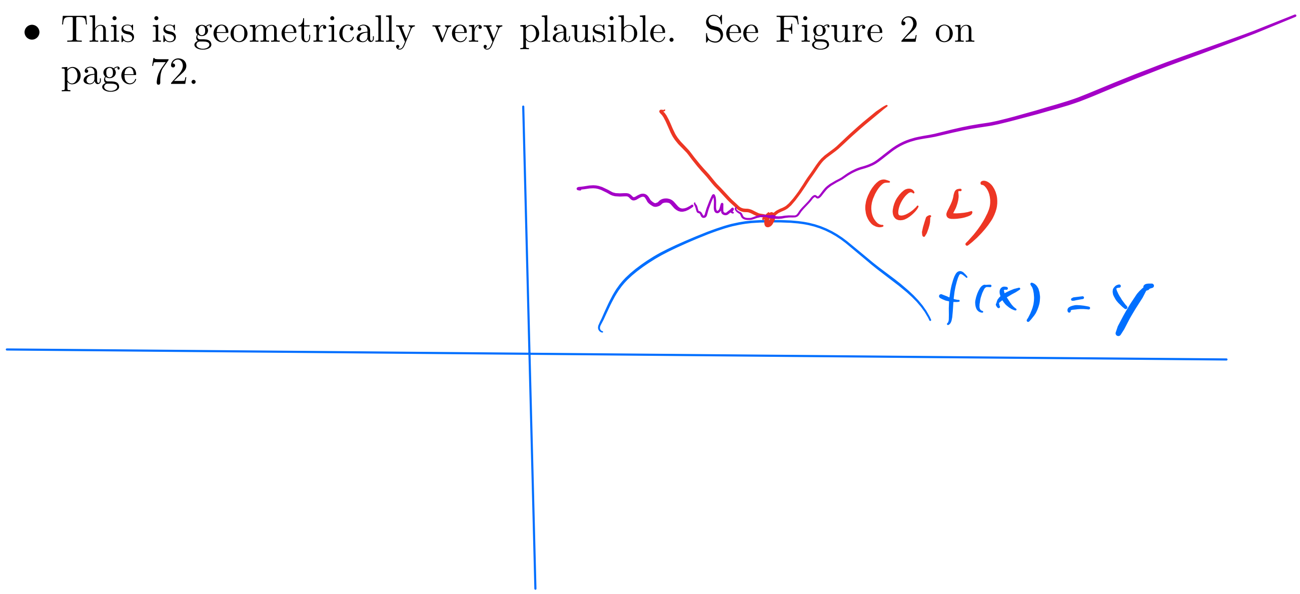
for all x near c except possibly *at* $x = c$. Also assume that

$$\lim_{x \longrightarrow c} f(x) = \lim_{x \longrightarrow c} h(x) = L$$

Then

$$\lim g(x) = L$$

- This is geometrically very plausible. See Figure 2 on page 72.



- Exercise for the ambitious: Prove the Squeeze Theorem using the $\epsilon - \delta$ definition of limits.

- Example:

$$\lim_{x \longrightarrow 0} x \sin \frac{1}{x} = 0.$$

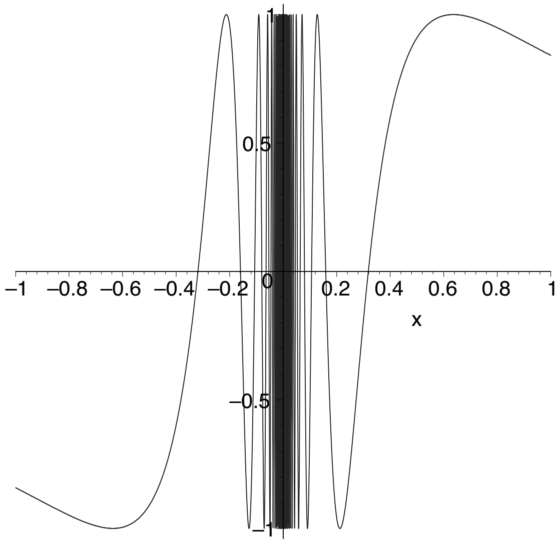


Figure 1. Graph of $y = \sin \frac{1}{x}$.

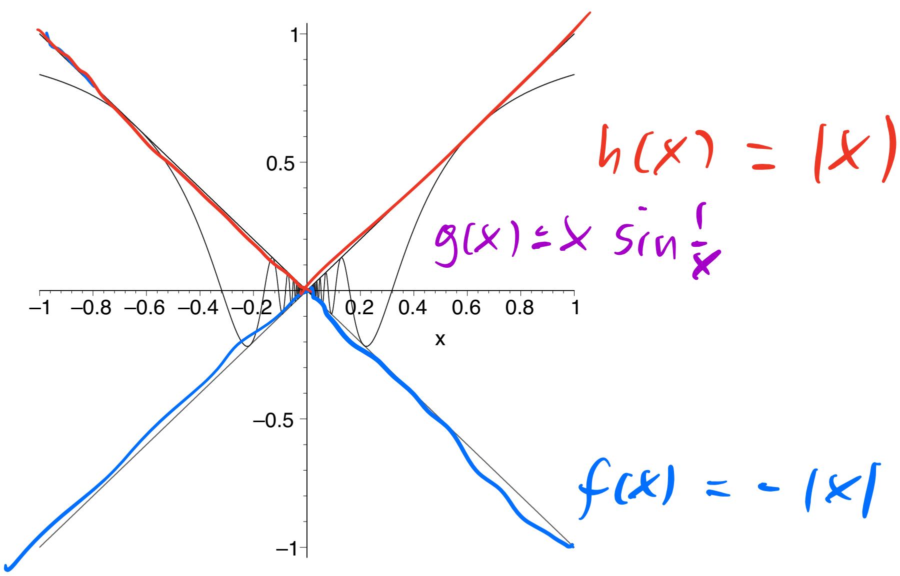


Figure 2. Graph of $y = x \sin \frac{1}{x}$ and $y = \pm|x|$.

One Sided Limits

- the limit properties we have discussed so far also apply to one sided limits.
- Examples:
- Recall

  $[\![x]\!]$ = the greatest integer $\leq x$.
- For example:

$[\![2]\!] = 2$

$$\lim_{x \longrightarrow 2^-} [\![x]\!]^2 + 1 = 2$$

$\lim_{x \to 2^-} [\![x]\!] = 1$

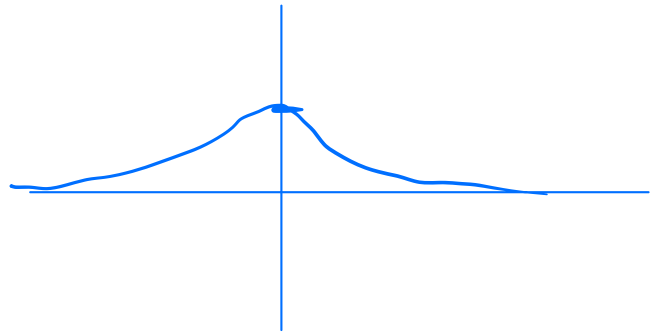
$$\lim_{x \longrightarrow 2^+} [\![x]\!]^2 + 1 = 5$$

$\lim_{x \to 2^+} [\![x]\!] = 2$

Limits and Infinity

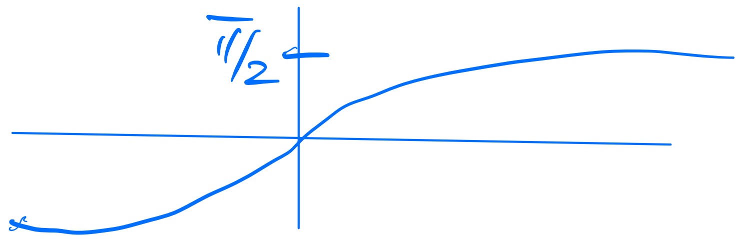
- There are also the concepts of "*limits at infinity*" and "*infinite limits*".
- Some examples:

$$\lim_{x \longrightarrow \infty} \frac{1}{1+x^2} = 0$$



$$\lim_{x \longrightarrow \infty} \arctan x = \frac{\pi}{2}$$

$$\lim_{x \longrightarrow -\infty} \arctan x = -\frac{\pi}{2}$$



- The next two examples are written as "infinite" limits, but actually are examples of non-existent limits.

$$\lim_{x \longrightarrow 1^+} \frac{x}{x-1} = \infty$$

$$\lim_{x \longrightarrow 1^-} \frac{x}{x-1} = -\infty$$

- Also note that

$$\lim_{x \longrightarrow \infty} \frac{x}{x-1} = 1$$

$$\lim_{x \longrightarrow 0} \frac{1}{x^2} = \infty$$

More Examples

$$\lim_{x \longrightarrow -\infty} \frac{1}{1+x^2} = \quad 0$$

$$\lim_{x \longrightarrow \infty} \frac{x^3+3x^2-\pi x+4}{3x^3+2x^2-4x+1} = \quad \frac{1}{3}$$

$$\lim_{x \longrightarrow \infty} \frac{x+\ln x}{x} = \quad 1$$

- View ahead to Math 1220 (Ex. 4, page 74), limits of sequences: Suppose

$$a_n = \sqrt{\frac{n+1}{n+2}}, \qquad n = 1, 2, 3, \ldots$$

Then

$$\lim_{n \longrightarrow \infty} a_n = \lim_{n \longrightarrow \infty} \sqrt{\frac{n+1}{n+2}} = \quad 1$$

Yet more Examples

$$\lim_{x \longrightarrow 2^+} \frac{x-1}{x-2} = \infty$$

$$\lim_{x \longrightarrow 2^-} \frac{x-1}{x-2} = -\infty$$

$$\lim_{x \longrightarrow 1} \frac{x-1}{x-2} = 0$$

$$\lim_{x \longrightarrow \infty} \frac{x^2}{2^x} = 0$$

$$\lim_{x \longrightarrow -\infty} \frac{x^2}{2^x} = \infty$$

$$\lim_{n \longrightarrow \infty} \sum_{k=0}^{n} \frac{1}{2^k} = 2$$

Yet More Examples

$$\lim_{x\longrightarrow 1} \frac{x^2+6x-7}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+7)}{x-1} = \lim_{x\to 1} x+7 = 8$$

$$\lim_{x\longrightarrow -7} \frac{x^2+6x-7}{x-1} = 0$$

$$\lim_{x\longrightarrow 0} \frac{x^2+6x-7}{x-1} = 7$$

$$\lim_{x\longrightarrow 4} \frac{x-4}{\sqrt{x}-2} =$$

$$\lim_{x\longrightarrow 3} \frac{x-3}{x+3} =$$

$$\lim_{x\longrightarrow 0} \frac{\sin(2x)}{x} =$$

exercises