Math 1210-23

Notes of 4/16-17/24

Review

- The remainder of the semester will consist of review. You want to be well prepared for the final exam, so make the most of what's left of this semester.
- Today and tomorrow I will go over the subject matter.
- HW 14 is meant to be a review of the semester. It is open now and closes April 24.

Math 1210 Summary

These Notes are numbered for easy reference.

1. As usual, this list is not complete, and it is not self contained. The individual items should stir your memory of things we discussed and things you did during the semester. If you draw a blank for any item that's a solid indication that you should review the relevant material before the final exam.

2. Prerequisites: know arithmetic, algebra, geometry, cartesian coordinates, functions, trigonometry, the difference between an simplifying an expression and an solving an equation.

3. **Major procedure**: come up with a concept, make it precise, derive its properties, and then use the properties to work with the concept.

4. We applied this procedure to three major concepts: **limits**, **derivatives**, and **integrals**.

5. Limits. $\lim_{x \longrightarrow c} f(x) = L$ means that for all $\epsilon > 0$ there is a $\delta > 0$ such that $0 < |x - c| < \delta$ implies that $|f(x) - c| < \epsilon$.

6. Most functions we deal with are **continuous**, i.e.,

$$\lim_{x \longrightarrow c} f(x) = f(c) \tag{1}$$

for all c in the domain of f.

7. If we have an expression that is undefined at a point, then manipulate it to get an expression that has the same value as the original everywhere and that can be evaluated at that point. We can do this because of the **Fundamental Limit Theorem** that states facts like that the limit of the sum, difference, product, or quotient is the sum, difference, product, or quotient of the limits.

8. Example:

$$\begin{aligned}
\lim_{h \longrightarrow 0} \frac{(x+h)^2 - x^2}{h} &= \lim_{h \longrightarrow 0} \frac{x^2 + 2hx + h^2 - x^2}{h} \\
&= \lim_{h \longrightarrow 0} \frac{2hx + h^2}{h} \\
&= \lim_{h \longrightarrow 0} (2x + h) \\
&= 2x.
\end{aligned} \tag{2}$$

9. Derivative

$$\begin{aligned}
f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x} f(x) \\
&= \lim_{h \longrightarrow 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{y \longrightarrow x} \frac{f(y) - f(x)}{y - x}
\end{aligned} \tag{3}$$

10. Geometrically: the slope of the tangent is the limit of the slopes of the secants, see Figure 1.

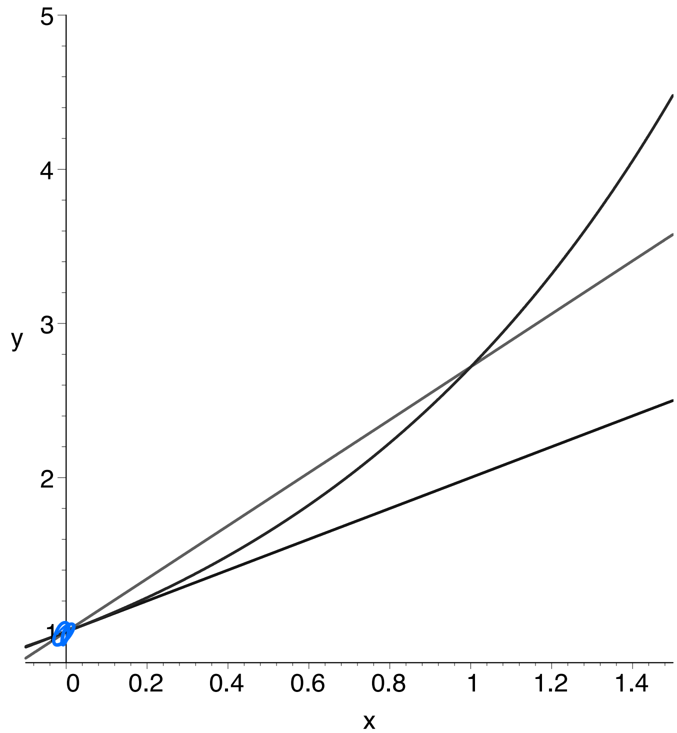

Figure 1. Definition of Derivative.

11. Example: Derivative of

$$f(x) = \frac{1}{2x+1}.$$

$$\frac{d}{dx} f(x) = \frac{d}{dx}(2x+1)^{-1} = -(2x+1)^{-2} \cdot 2$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{2(x+h)+1} - \frac{1}{2x+1}}{h}$$

$$= \lim \frac{\frac{2x+1 - (2x+2h+1)}{(2x+1)(2x+2h+1)}}{h} = \lim \frac{-2h}{h(2x+1)(2x+2h+1)}$$

$$= \lim_{h \to 0} \frac{-2}{(2x+1)(2x+2h+1)} = -\frac{2}{(2x+1)^2}$$

12. However, to compute derivatives we usually apply their properties, i.e., **Differentiation Rules**

$\frac{\mathrm{d}}{\mathrm{d}x}x^r$	$=$	rx^{r-1}	**Power Rule**
$(f+g)'$	$=$	$f'+g'$	**Sum Rule**
$(f-g)'$	$=$	$f'-g'$	**Difference Rule**
$(kf)'$	$=$	kf'	**Constant Multiple Rule**
$\frac{\mathrm{d}}{\mathrm{d}x}\sin x$	$=$	$\cos x$	**Sine Rule**
$\frac{\mathrm{d}}{\mathrm{d}x}\cos x$	$=$	$-\sin x$	**Cosine Rule**
$(uv)'$	$=$	$u'v+uv'$	**Product Rule**
$\left(\frac{u}{v}\right)'$	$=$	$\frac{u'v-uv'}{v^2}$	**Quotient Rule**
$\frac{\mathrm{d}}{\mathrm{d}x}f\big(g(x)\big)$	$=$	$f'\big(g(x)\big)g'(x)$	**Chain Rule**

(4)

Examples

13. Compute the derivatives of:

$f(x) = x^2 + 2x + 3$ $\quad f'(x) = 2x + 2$

$f(x) = x^3 - 2x^2 + x - 2$

$f(x) = (x^2+1)(x-2)$ (do it two ways)

$f'(x) = 3x^2 - 4x + 1$

$= 2x(x-2) + (x^2+1)\cdot 1 = 2x^2 - 4x + x^2 + 1$

$= 3x^2 - 4x + 1$

$f(x) = \cos x^2 - \cos^2 x$

$= -(\sin x^2)\, 2x - 2\cos x(-\sin x)$

$f(x) = \frac{\cos x^2}{\cos^2 x}$

$f'(x) = \frac{(-\sin x^2)\cdot 2x(\cos^2 x) + \cos x^2\, 2\cos x\,(+\sin x)}{\cos^4 x}$

$f(x) = \frac{1}{x^2+1}$

$f(x) = \sin^2 x^x + \cos^2 x^x$

$f'(x) = 0$

$f(x) = \sin\big(\sin(x)\big)$

$f'(x) = \cos(\sin(x))\cdot\cos x$

$f(x) = \frac{x\sin x^2}{1+\sin^2 x}.$

$f'(x) = \frac{(\sin x^2 + x(\cos x^2)2x)(1+\sin^2 x) - x\sin x^2\, 2\sin x\cos x}{(1+\sin^2 x)^2}$

14. Differentiation can be repeated, giving rise to higher derivatives, for example:

$$\begin{aligned} f(x) &= x^6, \\ f'(x) &= 6x^5, \\ f''(x) &= 30x^4, \\ f'''(x) &= 120x^3, \\ f^{(4)}(x) &= 360x^2, \\ f^{(5)}(x) &= 720x, \\ f^{(6)}(x) &= 720, \\ f^{(7)}(x) &= 0. \end{aligned} \tag{5}$$

$\int 0\,dx = C'$

$\int C'\,dx = C'x + C_1$

$\int Cx + C_1\,dx = \frac{C}{2}x^2 + C_1 x + C_2$

15. Differentiating a polynomial reduces its degree by 1.

16. In general, a function f is a polynomial of degree up to n if and only if the $(n+1)$-th derivative of f is everywhere zero.

17. **Newton's Method**, see Figure 2, can be used to find a root of a function f, i.e., a solution of the equation $f(x) = 0$. The basic idea is to construct a sequence where each term is the x-intercept to the tangent at the point corresponding to the previous term. You want to understand the formula

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}. \tag{6}$$

This means you can derive it and apply it.

$f(x)=0$

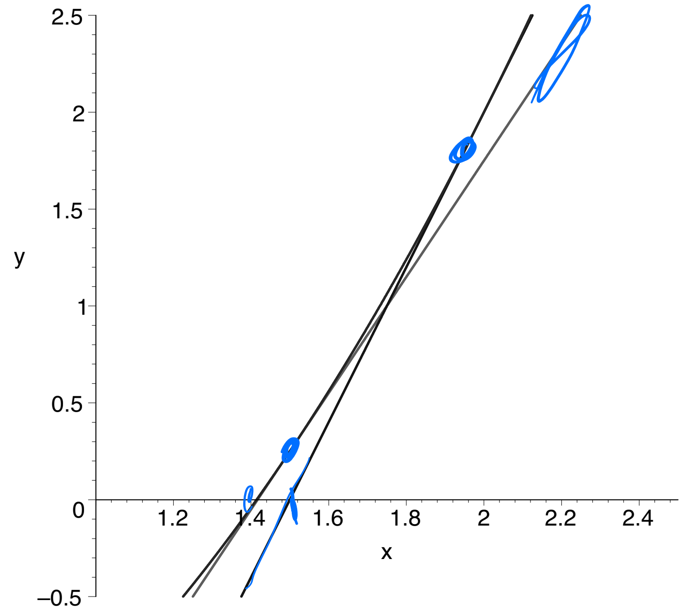

Figure 2. Newton's Method.

18. Differentiation can be done **implicitly**, for example, thinking of y as a function of x, we get

$y = y(x)$

$$x^2+y^2 = 1 \quad \Longrightarrow \quad 2x+2yy' = 0 \quad \Longrightarrow \quad y' = -\frac{x}{y} \tag{7}$$

19. Example: Consider the graph of the equation

$$x^2 + y^2 = r^2$$

and suppose that $P = (x_0, y_0)$ is a point on the graph. Compute the slope m of the tangent line at the point P. Your answer will be in terms of r, x_0, and possibly y_0.

4/16/24

20. Implicit Differentiation occurs frequently in **Related Rates Problems**: Write one or

$x_0 \qquad f(x) = 0$

$$T(x) = f(x_0) + f'(x_0)(x - x_0) = 0$$

$$T(x_0) = f(x_0)$$

$$T'(x) = f'(x_0)$$

$$f'(x_0)(x - x_0) = -f(x_0)$$

$$x - x_0 = -\frac{f(x_0)}{f'(x_0)}$$

$$x = x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

more equations that hold at all time, differentiate, obtain equations that involve rates (derivatives), solve for what you want to know.

21. Example: Suppose you cause an explosion in space (so there are no effects due to gravity or interaction with air). Your explosion produces a spherical fireball whose volume increases at a constant rate S. You wonder how fast the radius R of your fireball is increasing when its radius equals R. In other words, you want to compute $R'(R) =$

22. Example: You are at a distance D from the launch point of a hot air balloon. The balloon is rising vertically upwards at a velocity v. How fast is the angle of elevation of the balloon changing at your viewpoint when the height of the balloon is h miles?

23. Differentials: The change in a function value is approximately equal to the change in the independent variable, multiplied with the derivative.

$$\Delta y \approx \mathrm{d}y = f'(x)\mathrm{d}x = f'(x)\Delta x. \qquad (8)$$

24. Example: Suppose you are manufacturing spherical steel balls and you are concerned about how an error of p percent in the radius r will affect the volume. Compute the percentage error E in the volume. Your answer will be in terms of p and r. (We did this calculation in one of our review sessions, using differentials.)

25. Example 5, page 145, textbook. Poiseuille's Law for blood flow says that the volume flowing through an artery in a fixed time intervals is proportional to the fourth power of the radius. Estimate the increase in the radius required to increase the blood flow by 50 percent.

26. Source of word problems: the derivative of position is velocity, the derivative of velocity is acceleration.

27. Optimization: Minima and maxima can occur only at **critical points**:

- end points of intervals

- singular points (where the derivative does not exist)

- stationary points, where the derivative is zero.

28. The key to finding extreme values is to identify critical points and examine what happens there. Usually there are no singular points, and it is clear what happens at the endpoints. So typically you find stationary points.

29. Differentiate, set to zero, and solve.

30. Example: Suppose

$$f(x) = 20x - 4x^2$$

on the interval $[2, 4]$. Identify the critical points, and find the absolute maximum and minimum values. Show them as indicated below:

31. Example: You drill a hole of radius r along the axis of a sphere of radius R. This gives you a ring enclosing a cylinder. What value of r maximizes the volume of that cylinder?

32. Example: Suppose you are designing cylindrical cans with a given volume V. You want to minimize the amount of metal used for your can. Thus you want to pick the radius r and the height h of the can such that the surface area of the can (the combined area of the top and the bottom and the cylindrical wall of the can) is minimized. What is the *shape*, i.e., the ratio

$$s = \frac{\textbf{diameter}}{\textbf{height}} = \frac{2r}{h}$$

of that can?

33. If $f'(x)$ is positive (negative) in some interval then f is increasing (decreasing) in that interval.

34. If $f''(x)$ is positive (negative) in some interval then f is concave up (down) in that interval.

35. If $f'(x) = 0$ and the first derivative changes sign from positive to negative we must have a local minimum,

36. In particular, if $f'(x) = 0$ and $f''(x) > 0$ then $f(x)$ is a local minimum.

37. A similar statement applies to local maxima, of course.

38. A **point of inflection** is a point on the graph where the second derivative changes sign.

39. You should be able to draw graphs of functions using many sources of information, for example symmetry, singularities, asymptotes, and first and second derivatives. It's only rarely appropriate simply to compute a large number of points and plot them in a coordinate system.

40. Example: Discuss the shape of the graph of

$$f(x) = x(x+1)^3$$

41. Example: Discuss the shape of the graph of

$$f(x) = \frac{x^2+1}{(x+1)^2}$$

42. Rational Functions have asymptotes, vertical ones where the denominator is zero, the x-axis if the degree of the denominator exceeds that of the numerator, horizontal ones if the degrees of numerator and denominator are the same, and slanted ones if the degree of the numerator exceeds that of the denominator by 1.

43. The **Mean Value Theorem for derivatives**: If f is differentiable in (a,b) and continuous in $[a,b]$ then there is a point c in (a,b) such that

$$f(b) - f(a) = f'(c)(b-a). \tag{9}$$

44. **Differential Equations**: Equations that involve a function and some of its derivatives. Usually the goal is to find the function.

45. **Antiderivatives**. F is an antiderivative of f if $F' = f$. An integrable function f has infinitely many antiderivatives. Any two antiderivatives differ only by a constant.

46. The integration cannot may be determined by side conditions.

47. Suppose the function $v(t) = 2\sin t + 3t^2$ represents the velocity (in meters per second) of an object moving along the number line at time t. If the object is located at $x = 5$ meters when $t = 0$, find a formula for $x(t)$, the object's position as a function of time.

48. Example: You are braking at a deceleration of 25 feet per second squared and come to a stop after 5 seconds. How fast were you traveling when you started braking, and how far did you travel while braking?

49. **Indefinite Integrals**.

$$\int f(x)\mathrm{d}x = F(x) + C \qquad (10)$$

where f is the **integrand**, $F' = f$, and C is the **integration constant**. F is an **antiderivative** of f. The indefinite integral is the set of all antiderivatives. The value of the integration constant may be determined by a side condition.

50. Definite Integrals as limits of Riemann Sums:

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\longrightarrow\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where

$$\Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i\Delta x \qquad (11)$$

51. When computing Riemann Sums, some special sum rules are useful:

$$\begin{aligned} \sum_{i=1}^{n} 1 &= n, \\ \sum_{i=1}^{n} i &= \frac{n(n+1)}{2}, \\ \sum_{i=1}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6} \end{aligned} \quad . \qquad (12)$$

52. Example: Compute

$$S = \sum_{i=10}^{100} i.$$

53. If $f(x)$ is non-negative everywhere in $[a, b]$ then $\int_a^b f(x)\mathrm{d}x$ is the area of the region bounded by the x-axis, the graph of f, and the vertical lines $x = a$ and $x = b$.

54. We use Riemann sums to recognize as a definite integer what we are trying to compute. However, we usually compute definite integrals by one version of the **Fundamental Theorem of Calculus:**

$$\boxed{\int_a^b f(x)\mathrm{d}x = F(b) - F(a) \qquad \text{where} \qquad F' = f.} \tag{13}$$

55. Some integrals can be computed without knowing an antiderivative. We discussed in particular:

$$\int_{-c}^{c} f(x)\mathrm{d}x = 0 \quad \text{if } f \text{ is odd,} \tag{14}$$

$$\int_a^{a+2\pi} \sin^2 x\mathrm{d}x = \int_a^{a+2\pi} \cos^2 x\mathrm{d}x = \pi, \tag{15}$$

and

$$\int_{-r}^{r} \sqrt{r^2 - x^2}\mathrm{d}x = \frac{\pi r^2}{2}. \tag{16}$$

56. Every differentiation rule is also an integration rule.

57. Integration by Substitution is the inverse process of the chain rule.

58. Example: Evaluate the definite integral

$$\int_0^4 3x\sqrt{x^2 + 9}\mathrm{d}x =$$

59. The **Mean Value Theorem for Integrals**: There is a point c in $[a, b]$ such that

$$\int_a^b f(x)\mathrm{d}x = f(c)(b-a). \tag{17}$$

60. You can be casual when computing an antiderivative, once you have it check it by differentiation.

61. We used definite integrals to solve the following problems:

- Computation of areas

62. Example: Find the area enclosed by the curves $y = x^4$ and $x = y^2$.

- Computation of volumes by integrating the area of the cross section, using the methods of slabs, disks, washers, or shells.

63. Example By computing the volume of a solid of revolution, show that the volume of a right circular cone of height H and radius R equals

$$V = \frac{\pi R^2 H}{3}.$$

- Computation of the length of a plane curve.

- Computation of the surface area of a solid of revolution

- Computation of Work.

- Computation of the center of mass.

64. The limits of integration may depend on a variable. We can differentiate with respect to that variable without actually computing an antiderivative:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_{L(x)}^{U(x)} f(t)\mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}x}\Big(F\big(U(x)\big) - F\big(L(x)\big)\Big)\\ &= f(U(x))U'(x) - f(L(x))L'(x)\end{aligned} \tag{18}$$

where, as usual, F is any antiderivative of f.

65. A special case of that formula is this version of the Fundamental Theorem of Calculus:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\int_{a}^{x} f(t)\mathrm{d}t = f(x).} \tag{19}$$

66. The Fundamental Theorem of Calculus says that differentiation and integration are inverse processes of each other.

67. Example Evaluate the integral

$$\int_{-2}^{2} \sqrt{4 - x^2} \mathrm{d}x =$$

68. Example Evaluate the integral

$$\int_{-\pi}^{\pi} \frac{\sin x}{1 + x^2} \mathrm{d}x =$$

69. Example Compute $\frac{\mathrm{d}}{\mathrm{d}x} \int_{\pi}^{x^2} \frac{1}{1+t^2} \mathrm{d}t =$

70. Following is a list of some words and phrases, listed in alphabetical order, that you should be able to define, use, and understand.

acceleration, antiderivative, asymptote, base, chain rule, concave down, concave up, constant, continuity, critical points, cubic, decreasing function, definite integral, degree of a polynomial, denominator, dependent variable, derivative, differential, differential equation, domain, even function, equation, exponent, expression, first derivative, function, Fundamental Limit Theorem, Fundamental Theorem of Calculus, graph (of a function or an equation), implicit differentiation, increasing function, indefinite integral, independent variable, inflection point, integrand, integration constant, integration variable, Leibniz notation, limit, limits of integration (upper and lower), linear, Mean Value Theorem for derivatives, Mean Value Theorem for integrals, method of disks, method of shells, method of slabs, method of washers, Newton's method, numerator, odd functions, points of inflection, polynomial, position, power rule, power, product rule, quadratic, quartic, quintic, quotient rule, radical, range, rational function, related rates, Riemann Sum, secant, second derivative, singular point, solid of revolution, stationary point, sum rules, tangent, velocity, work.

71. Classes building on this one. You want to take them in the given sequence.

1220 More differentiation and integration rules (particularly exponentials, logarithms, inverse trig functions, integration by parts, logarithmic differentiation), more applications, indeterminate expressions, improper integrals, sequences and series (particularly their convergence, power series, Taylor series).

2210 Calculus of several variables.

2270 Linear Algebra

2280 Differential Equations

2250 For engineering students, covers both linear algebra and differential equations. Computer science may require 2270 instead of 2250 (talk to your CS advisor).