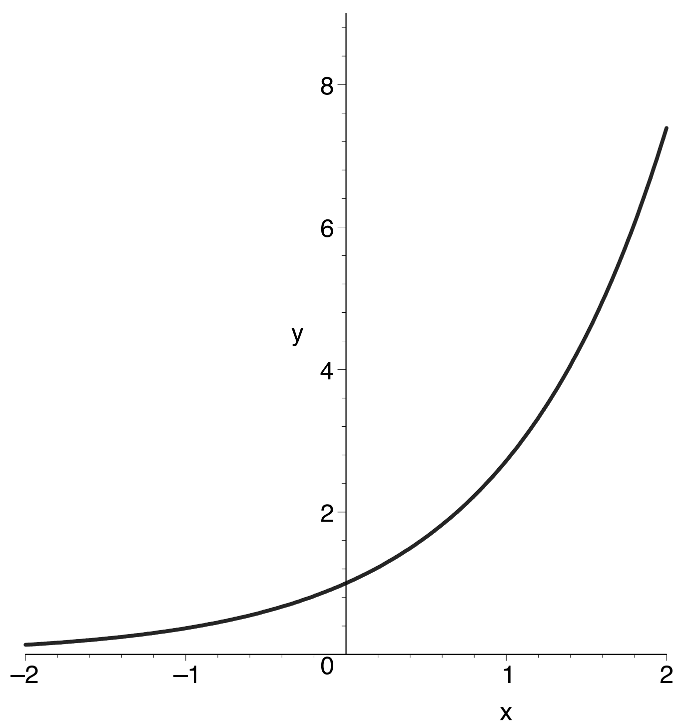
Math 1210-23

Notes of 4/15/24

- Today: Exponentials and Logarithms.
- For some this will be the last Calculus class you take. If so, then you will have missed one major item: the derivative of the exponential.
- This will be the first topic in Math 1220.
- Let's briefly talk about it today.
- We'll also see a spectacular application of Calculus.



$f(x) = e^x$

$e = 2.71828\ldots$

Figure 1. The Exponential.

$$\frac{d}{dx} e^{kx} = k e^{kx}$$

- Major fact: The exponential equals its own derivative.

$$\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x.$$

- By the same token:

$$\int e^x \mathrm{d}x = e^x + C.$$

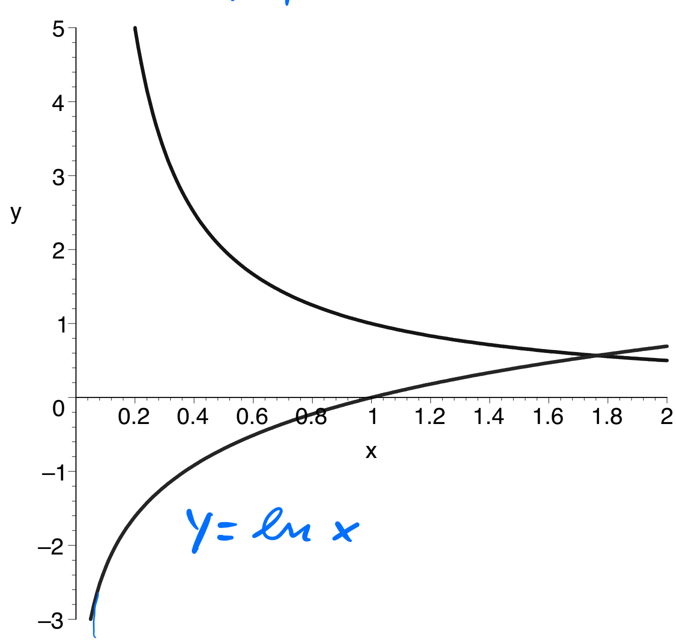
- Makes geometric sense.
- It is also true that

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{x} \quad \text{and} \quad \int \frac{1}{x} = \ln x + C$$

$x^{-1} = \frac{1}{x}$

$p \neq -1 \quad \int x^p dx = \frac{x^{p+1}}{p+1} + C$

$\exp(x) = e^x$



$\ln e^x = x$

Figure 2. Derivative of the Logarithm.

- This also makes geometric sense.
- For other exponentials and logarithms we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}a^x &= \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{(\ln a)}\right)^x = \frac{d}{dx} e^{x \ln a} \\ &= \ln a e^{x \ln a} \\ &= (\ln a) a^x\end{aligned}$$

and, similarly,

$$\frac{\mathrm{d}}{\mathrm{d}x}\log_a(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\ln x}{\ln a} = \frac{1}{x\ln a}.$$

Differentiation Examples

$$\frac{\mathrm{d}}{\mathrm{d}x} e^{x^2} = 2x\, e^{x^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(e^x\right)^2 = 2e^x e^x = 2e^{2x} = 2\left(e^x\right)^2$$

$$\frac{\mathrm{d}}{\mathrm{d}x} x^x =$$

$$y = x^x \qquad \ln x^x = \ln y = x \ln x$$

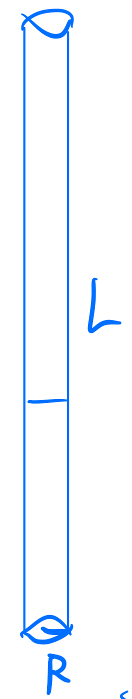
$$\frac{y'}{y} = (\ln x) + x\frac{1}{x} = 1 + \ln x$$

$$y' = y(1 + \ln x)$$

$$= x^x(1 + \ln x)$$

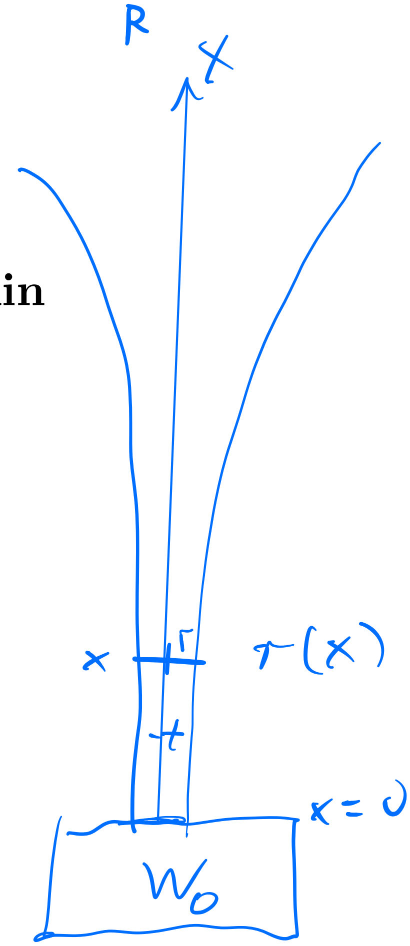
Some Rope

This may be my favorite Calculus problem. It certainly is a spectacular application of Calculus! We will put together much of what we learned: **differentiation, integration, differential equations, volume calculations, the fundamental theorem of Calculus!**



Suppose you want to build a rope that can reach a very great depth. The trouble with an ordinary, cylindrical, rope is that if it is too long it will break under its own weight. The length at which it would break is independent of its radius, since as the radius increases so does the weight of the rope. The two processes exactly balance. Let's look a little more closely. The rope will break when the strain at some point exceeds a critical value that depends on the material of which the rope is made. Thus the rope will break when

$$\textbf{strain} = \frac{\textbf{weight}}{\textbf{area of cross-section}} > \textbf{critical strain} \tag{1}$$



Now consider an ordinary rope of length L and radius r. The weight of that rope equals

$$w = \pi r^2 L \delta \tag{2}$$

where δ is the specific weight of that rope. The area of the cross section is

$$A = \pi r^2 \tag{3}$$

and so we obtain the condition

$$\frac{\pi r^2 L\delta}{\pi r^2} = L\delta > \gamma_0 \tag{4}$$

or

$$L > \frac{\gamma_0}{\delta} = L_0 \tag{5}$$

where γ_0 is the critical strain and the critical length L_0 is independent of the radius of the rope.

However, we can increase the depth a rope can reach beyond the critical length by *increasing the radius of the rope towards the top*! So suppose we let the radius of the rope be $r(x)$ where x is the length of the rope measured upwards from the bottom. The weight of the rope at a point x equals the volume of the rope above x multiplied with the specific weight w. We want to find the function $r(x)$ that makes the strain $\gamma(x)$ in the rope constant (and well below the critical value γ_0). The volume of the rope can be computed easily by considering it a solid of revolution. Let's also suppose that at the bottom of the rope it will carry a weight w_0.

Putting this information into mathematical terms we obtain the equation

$$\frac{\delta \int_0^x \pi r^2(t)\mathrm{d}t + w_0}{\pi r^2(x)} = \gamma \tag{6}$$

or

$r(t) = ?$

$$\delta \int_0^x \pi r^2(t)\mathrm{d}t + w_0 = \gamma \pi r^2(x) \tag{7}$$

Differentiating in (7) gives (by the fundamental theorem of Calculus)

$$\delta \cancel{\pi} r^2(x) = 2\cancel{\pi}\gamma r(x) r'(x). \tag{8}$$

Solving for $r'(x)$ gives the differential equation

$$r'(x) = \underbrace{\frac{\delta}{2\gamma}}_{k} r(x) \tag{9}$$

$$r' = k r$$

which has the solution

$$r(x) = e^{\frac{\delta}{2\gamma}x + C_1} = e^{\frac{\delta}{2\gamma}x + \ln C} = e^{C\frac{\delta}{2\gamma}x} = C_1 e^{\frac{\delta}{2\gamma}x}$$

$$r(x) = C \exp\left(\frac{\delta}{2\gamma}x\right). \tag{10}$$

To determine $C = r(0)$ we set $x = 0$ in (7) and obtain

$$w_0 = \gamma \pi r^2(0) \tag{11}$$

which gives

$$C_1 = r(0) = \sqrt{\frac{w_0}{\gamma\pi}}. \tag{12}$$

Hence

$$\boxed{r(x) = \sqrt{\frac{w_0}{\gamma\pi}} \exp\left(\frac{\delta}{2\gamma}x\right)} \tag{13}$$

which is the formula that we have been looking for!