Math 1210-23

Remaining Events

Fr 4/12 Center of Mass, 5.6 textbook
- — Study Session, Andy, after class
- — 2:00-2:50, Office Hours, Alex

Mo 4/15 Calculus of Exponentials and Logarithms, 6.1–5, textbook
- — 9:40-10:25, Peter, Office Hours, JWB 127
- — 12:55-1:45, Mia, Office Hours, LCB 322

Tu 4/16 Review
- — 11:50-12:40, Office Hours, Liza, LCB 322
- — 2:00-2:50 Study Session, Andy, MLi 1130

We 4/17 Review
- — 9:40-10:30, Study Session, Andy, LCB 215
- — 11:50-12:40, Office Hours, Liza, LCB 322

Th 4/18 Last Day of Labs

Fr 4/19 Review
- — 9:40-10:30, Office Hours, Peter, JWB 127
- — Study Session, Andy, after class
- — 2:00-2:50 Office Hours, Alex

Mo 4/22 Review
- — 9:40-10:25, Peter, Office Hours, JWB 127
- — 12:55-1:45, Mia, Office Hours, LCB 322

Tu 4/23 Review, Classes end

— 11:50-12:40, Office Hours, Liza, LCB 322

— 2:00-2:50 Study Session, Andy, MLi 1130

We, 4/24 Reading Day, no events

Th, 4/25 Study Session, Peter, 10:30am-11:55am, LCB 215

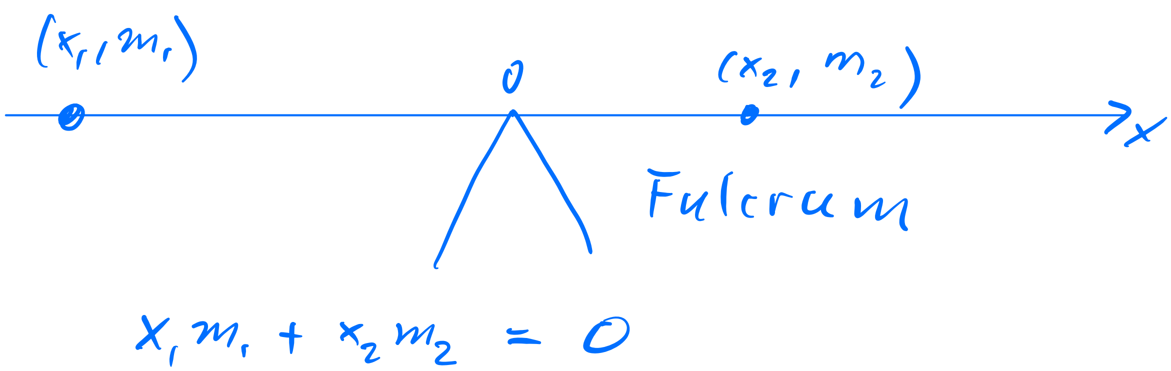
Fr, 4/26 Study Session, Peter, 10:30am-12:30pm, LCB 215

Mo, 4/29 10:30am-12:30pm, Final Exam, The End

Notes of 4/12/24

5.6 Center of Mass

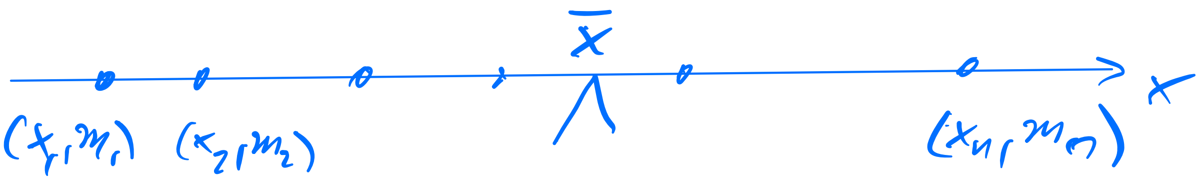
- Another application of integration.
- We start with a seesaw. x indicates location along the seesaw. The fulcrum is at $x = 0$ and we have two masses m_1 and m_2 at locations x_1 and x_2, respectively.



- Noting that x_1 is negative we see that the seesaw will be in equilibrium if

$$x_1 m_1 + x_2 m_2 = 0.$$

- Here is a modification of the previous problem. Suppose you have n masses m_1, m_2, …, m_n in locations x_1, x_2, …, x_n. How do you choose the location $\bar{x}$ of the fulcrum so that the seesaw is in equilibrium?



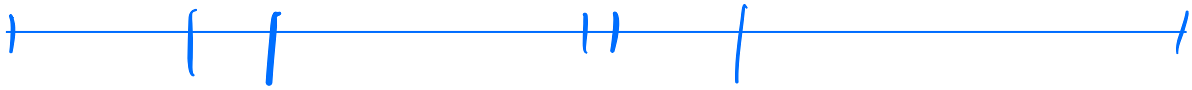
$$\sum_{i=1}^{n} = \sum$$

$$\sum (x_i - \bar{x}) m_i = 0 \Rightarrow \sum x_i m_i - \sum \bar{x} m_i = 0$$

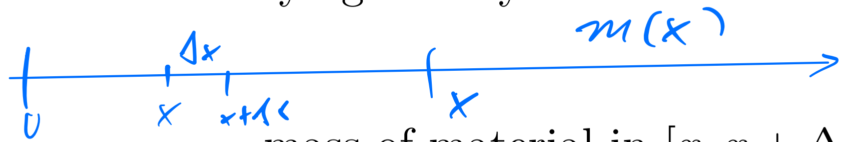
$$\sum x_i m_i = \sum \bar{x} m_i = \bar{x} \sum m_i$$

$$\bar{x} = \frac{\sum x_i m_i}{\sum m_i}$$

- You can see where this is going!



- What if we have a continuous mass distribution?
- Like a wire with varying density.



$$\text{density } \delta(x) = \lim_{\Delta x \longrightarrow 0} \frac{\text{mass of material in } [x, x + \Delta x]}{\Delta x}$$

- Could be caused by varying the thickness of the wire, for example.
- As many times before, the sums turn into integrals:

$$\bar{x} = \frac{\sum_{i=1}^n m_i x_i}{\sum_{i=1}^n m_i} \qquad \longrightarrow \qquad \boxed{\bar{x} = \frac{\displaystyle\int_a^b x\delta(x)\mathrm{d}x}{\displaystyle\int_a^b \delta(x)\mathrm{d}x}.}$$

Example 2: Suppose

$$\delta(x) = 3x^2, \qquad [a, b] = [0, 10]$$

- Expectations?

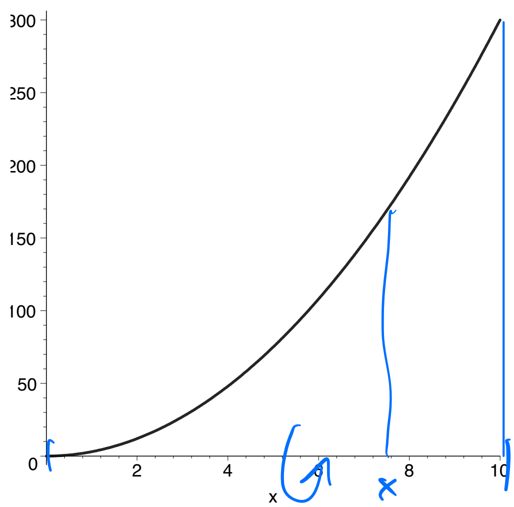


Figure 1. Example 2.

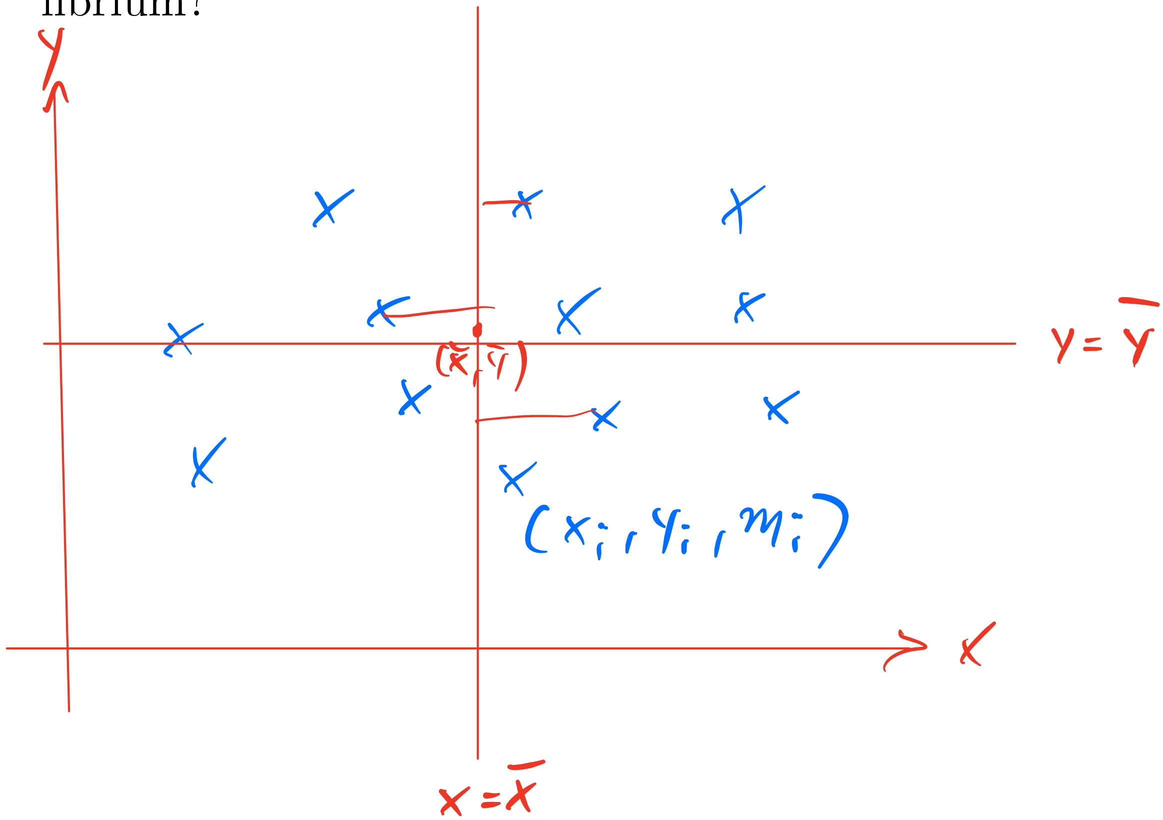
$$\bar{x} = \frac{\int_a^b x\delta(x)\,dx}{\int_a^b \delta(x)\,dx} = \frac{\int_0^{10} x\,3x^2\,dx}{\int_0^{10} 3x^2\,dx} = \frac{7500}{1000} = 7.5$$

$$\int_0^{10} 3x^2\,dx = \left[x^3\right]_0^{10} = 1000$$

$$\int_0^{10} 3x^3\,dx = \left[\frac{3x^4}{4}\right]_0^{10} = 7500$$

Point Masses in a plane

- Suppose we have point masses (x_i, y_i, m_i), $i = 1, 2, \ldots n$ on a massless platter. Where do we need to support the platter to keep it in equilibrium?



- The **center of mass** is given by

$$\bar{x} = \frac{\displaystyle\sum_{i=1}^{n} m_i x_i}{\displaystyle\sum_{i=1}^{n} m_i} \quad \text{and} \quad \bar{y} = \frac{\displaystyle\sum_{i=1}^{n} m_i y_i}{\displaystyle\sum_{i=1}^{n} m_i}$$

- Example: Suppose we have the three masses $(0, 1, 3)$, $(1, 0, 2)$, and $(0, 0, 1)$. Where is the center of mass?

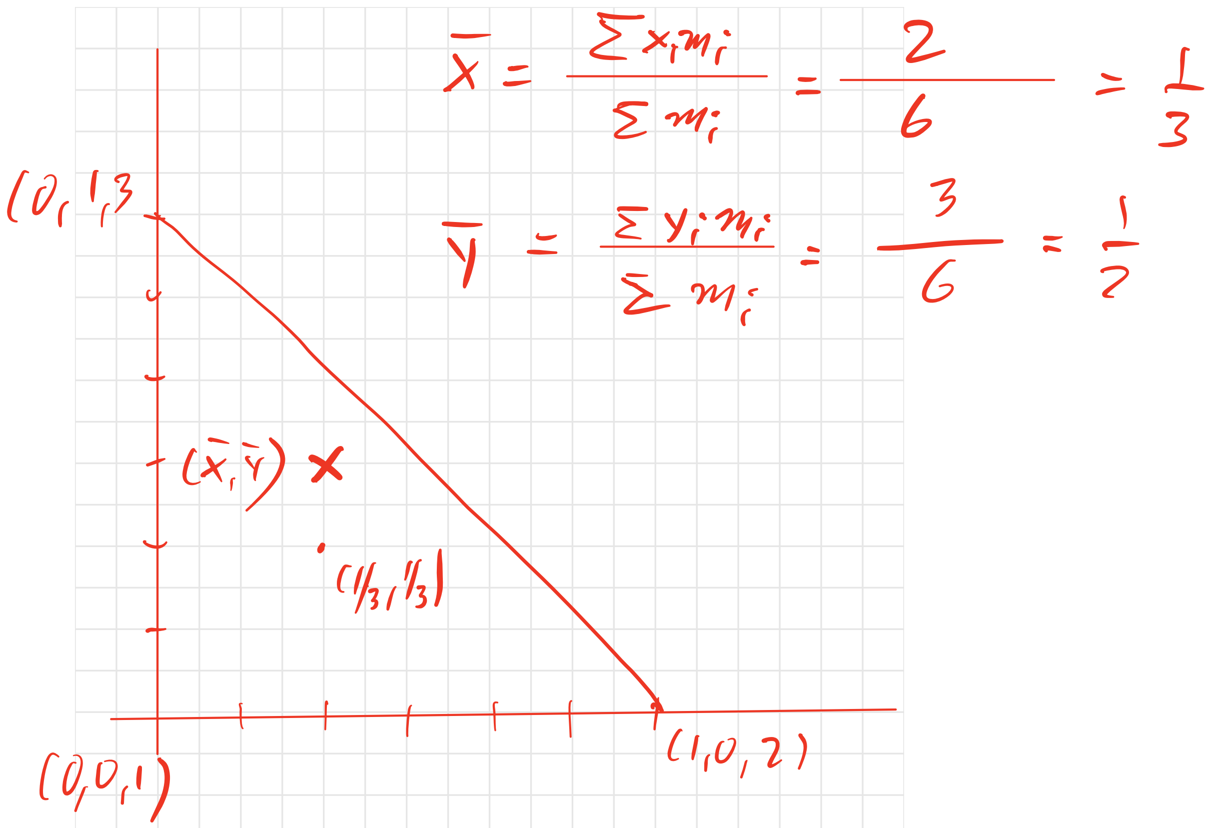


Figure 2. Three Points.

- Expectations?

- What is the continuous analog?
- Where do you support a thin region to keep it balanced?
- For simplicity let's assume the material is homogeneous:

$$\delta(x, y) = 1$$

- Suppose the region is bounded by two functions, $f(x)$ and $g(x)$ where $a \leq x \leq b$.

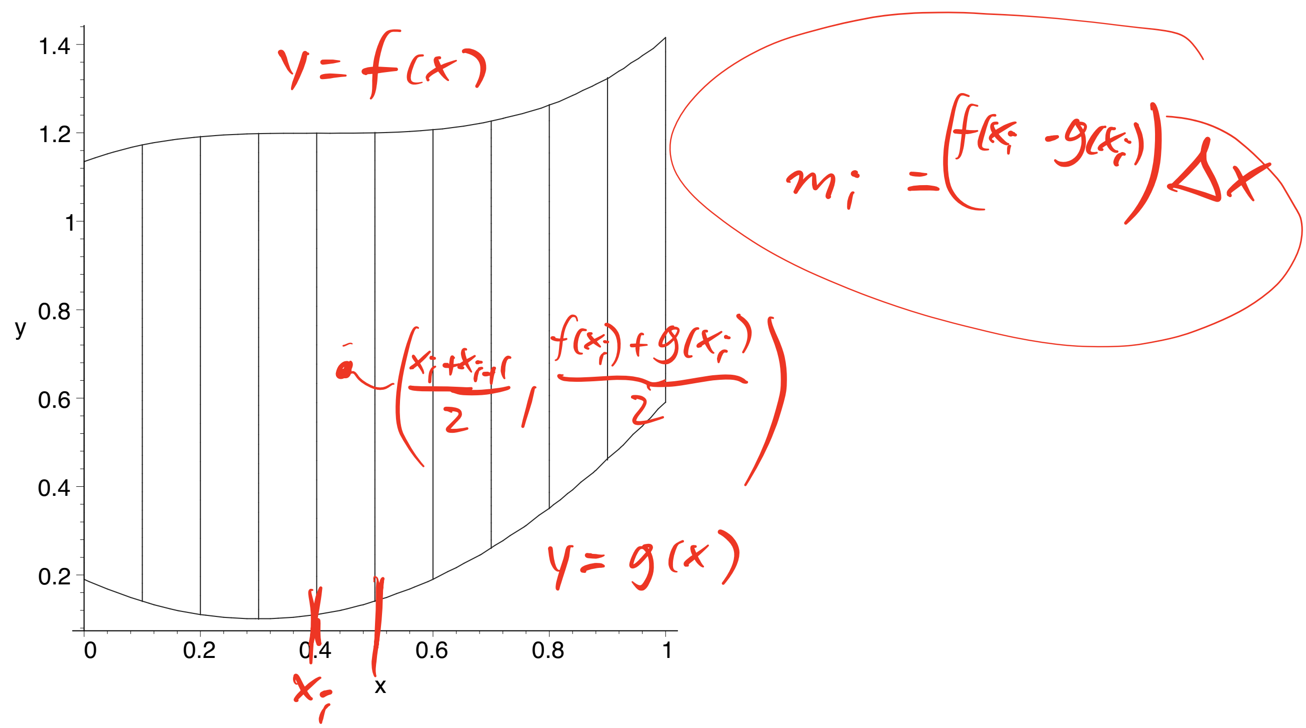


Figure 3. A Lamina.

- We divide the region into vertical slice and think of each slice as being approximately a rectangle. The mass of the rectangle is proportional to its area, and we think of that area as being concentrated in the center of the rectangle.

- The point formulas

$$\bar{x} = \frac{\displaystyle\sum_{i=1}^{n} m_i x_i}{\displaystyle\sum_{i=1}^{n} m_i} \quad \text{and} \quad \bar{y} = \frac{\displaystyle\sum_{i=1}^{n} m_i y_i}{\displaystyle\sum_{i=1}^{n} m_i}$$

turn into

$$\bar{x} = \frac{\displaystyle\int_a^b x\big(f(x) - g(x)\big)\mathrm{d}x}{\displaystyle\int_a^b \big(f(x) - g(x)\big)\mathrm{d}x}$$

and

$$\begin{aligned}\bar{y} &= \frac{\displaystyle\int_a^b \frac{1}{2}\big(f(x) + g(x)\big)\big(f(x) - g(x)\big)\mathrm{d}x}{\displaystyle\int_a^b \big(f(x) - g(x)\big)\mathrm{d}x} \\ &= \frac{\displaystyle\int_a^b \frac{1}{2}\big(f^2(x) - g^2(x)\big)\mathrm{d}x}{\displaystyle\int_a^b \big(f(x) - g(x)\big)\mathrm{d}x}\end{aligned}$$

- This is not intuitive!
- The formulas for $\bar{x}$ and $\bar{y}$ ought to be more symmetric.

- In Math 2210 we will learn about multiple integrals, and obtain symmetric formulas.
- For our purposes today, let's finish this off by looking at an example.
- Semi-circle, radius 1,

$$f(x) = \sqrt{1 - x^2} \quad \text{and} \quad g(x) = 0$$

$\bar{y} \approx \frac{1}{2}$

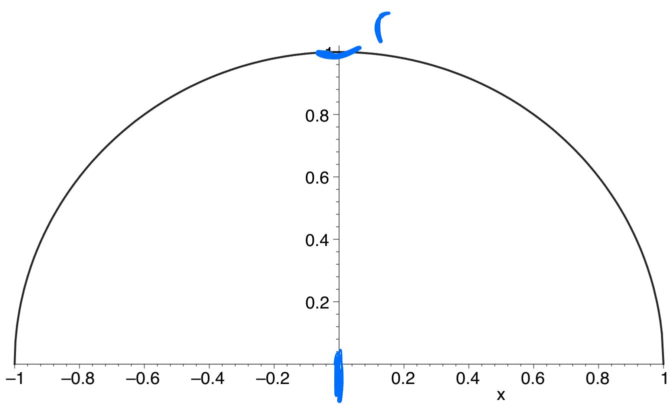


Figure 4. Semi-circle.

$\frac{1}{2}\left[x - \frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{2}\left(\frac{2}{3} + \frac{2}{3}\right) = \frac{2}{3}$

- Expectations?

$$\bar{y} = \frac{\frac{1}{2}\int_a^b \left(f^2(x) - g^2(x)\right) d}{\int_a^b f(x) - g(x)\, dx} = \frac{\frac{1}{2}\int_{-1}^{1} 1 - x^2\, dx}{\int_{-1}^{1} \sqrt{1 - x^2}\, dx}$$

$g(x) = 0$

$f(x) = \sqrt{1 - x^2}$

$\int_{-1}^{1} \sqrt{1-x^2}\,dx = \pi/2$

$$\bar{y} = \frac{2/3}{\pi/2} = \frac{4}{3\pi}$$

- (very) major omission this semester: Derivatives and integrals of exponentials and logarithms.
- Any guess?

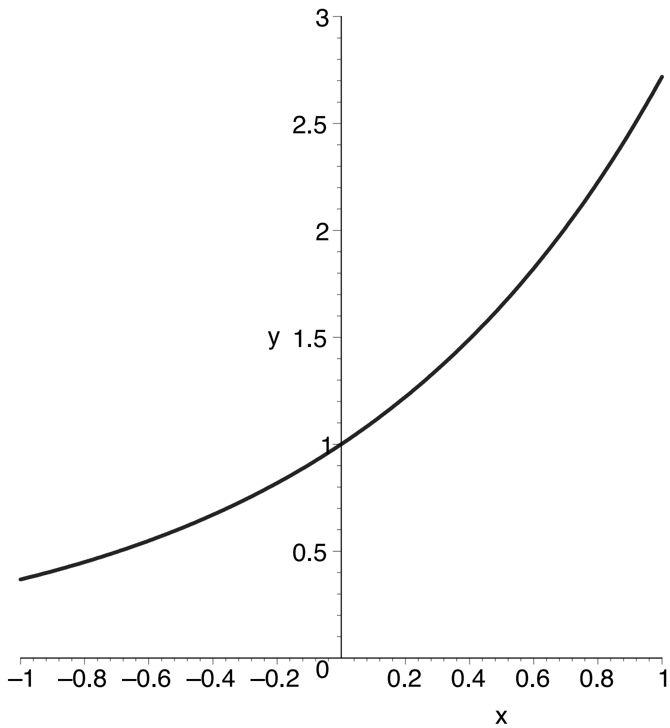


Figure 5. The Exponential.