Announcements

- Schedule of events now on Canvas. Link on Home Page.

1.3, 1.5, Working with Limits

- Recall Procedure:
  - Concept $\rightarrow$ Definition $\rightarrow$ Properties $\rightarrow$ Work

- Our Definition: We say that the limit of $f(x)$ as $x$ approaches $c$ equals $L$, or

$$
\lim_{x \to c} f(x) = L
$$

if for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$
0 < |x - c| < \delta \quad \implies \quad |f(x) - L| < \epsilon.
$$

- $\epsilon$ is the is the lower case Greek letter $epsilon$, and $\delta$ is the lower case Greek letter $delta$. 
Properties:

Main Limit Theorem

• See textbook, page 68.
• Let \( n \) be a positive integer, \( k \) a constant, and \( f \) and \( g \) functions that have limits at \( c \). Then:

1. \[ \lim_{x \to c} k = k. \]

2. \[ \lim_{x \to c} x = c. \]

3. \[ \lim_{x \to c} kf(x) = k \lim_{x \to c} f(x). \]

4. \[ \lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x). \]

5. \[ \lim_{x \to c} (f(x) - g(x)) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x). \]

6. \[ \lim_{x \to c} (f(x) \cdot g(x)) = \left( \lim_{x \to c} f(x) \right) \cdot \left( \lim_{x \to c} g(x) \right). \]

7. \[ \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} \text{ provided } \lim_{x \to c} g(x) \neq 0. \]

8. \[ \lim_{x \to c} (f(x))^n = \left( \lim_{x \to c} f(x) \right)^n. \]

9. \[ \lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)} \text{ provided } f(x) \geq 0 \text{ when } n \text{ is even.} \]
• Consequence of Main Limit Theorem:

\[
\lim_{{x \to c}} f(x) = f(c)
\]

if \( f \) is a polynomial or a rational function with a non-zero denominator at \( x = c \).

• Example

\[
\lim_{{x \to 1}} \frac{x + 4}{x^2 + 1} =
\]
• subtle point, and frequent source of errors: when combining two functions the limit may exist even if the individual limits do not.

• simple example:

\[ f(x) = \frac{1}{x} \quad \text{and} \quad g(x) = -\frac{1}{x} \]

What happens to \( f(x) + g(x) \) as \( x \) goes to zero? The individual limits do not exist, but the limit of the sum is zero.
• We proved item 4 of the main limit theorem in class.

• Proof of the other parts is in the textbook in section 1.3.

• Another major fact is

**The Squeeze Theorem.** Suppose \( f, g, \) and \( h \) are functions such that

\[
f(x) \leq g(x) \leq h(x)
\]

for all \( x \) near \( c \) except possibly at \( x = c \). Also assume that

\[
\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L
\]

Then

\[
\lim_{x \to c} g(x) = L
\]

• This is geometrically very plausible. See Figure 2 on page 72.

• Exercise for the ambitious: Prove the Squeeze Theorem using the \( \epsilon - \delta \) definition of limits.
Example:

\[
\lim_{x \to 0} x \sin \frac{1}{x} = 0.
\]

Figure 1. Graph of \( y = \sin \frac{1}{x} \).

Figure 2. Graph of \( y = x \sin \frac{1}{x} \) and \( y = \pm |x| \).
One Sided Limits

• the limit properties we have discussed so far also apply to one sided limits.

• Examples:

• Recall
  
  \[ [x] = \text{the greatest integer } \leq x. \]

• For example:

  \[
  \lim_{x \to 2^-} [x]^2 + 1 =
  \]

  \[
  \lim_{x \to 2^+} [x]^2 + 1 =
  \]
Limits and Infinity

• There are also the concepts of "limits at infinity" and "infinite limits".

• Some examples:

\[
\lim_{x \to \infty} \frac{1}{1 + x^2} =
\]

\[
\lim_{x \to \infty} \arctan x =
\]

\[
\lim_{x \to -\infty} \arctan x =
\]

• The next two examples are written as "infinite" limits, but actually are examples of non-existent limits.

\[
\lim_{x \to 1^+} \frac{x}{x - 1} =
\]

\[
\lim_{x \to 1^-} \frac{x}{x - 1} =
\]

• Also note that

\[
\lim_{x \to \infty} \frac{x}{x - 1} =
\]

\[
\lim_{x \to 0} \frac{1}{x^2} =
\]
More Examples

\[ \lim_{x \to -\infty} \frac{1}{1 + x^2} = \]

\[ \lim_{x \to \infty} \frac{x^3 + 3x^2 - \pi x + 4}{3x^3 + 2x^2 - 4x + 1} = \]

\[ \lim_{x \to \infty} \frac{x + \ln x}{x} = \]

- View ahead to Math 1220 (Ex. 4, page 74), limits of sequences: Suppose

\[ a_n = \sqrt{\frac{n + 1}{n + 2}}, \quad n = 1, 2, 3, \ldots \]

Then

\[ \lim_{n \to \infty} a_n = \lim_{n \to \infty} \sqrt{\frac{n + 1}{n + 2}} = \]
Yet more Examples

\[
\lim_{x \to 2^+} \frac{x - 1}{x - 2} = \\
\lim_{x \to 2^-} \frac{x - 1}{x - 2} = \\
\lim_{x \to 1} \frac{x - 1}{x - 2} = \\
\lim_{x \to \infty} \frac{x^2}{2^x} = \\
\lim_{x \to -\infty} \frac{x^2}{2^x} = \\
\lim_{n \to \infty} \sum_{k=0}^{n} \frac{1}{2^k} =
\]
Yet More Examples

\[ \lim_{x \to 1} \frac{x^2 + 6x - 7}{x - 1} = \]

\[ \lim_{x \to 7} \frac{x^2 + 6x - 7}{x - 1} = \]

\[ \lim_{x \to 0} \frac{x^2 + 6x - 7}{x - 1} = \]

\[ \lim_{x \to 4} \frac{x - 4}{\sqrt{x} - 2} = \]

\[ \lim_{x \to 3} \frac{x - 3}{x + 3} = \]

\[ \lim_{x \to 0} \frac{\sin(2x)}{x} = \]