Math 1210-23 Notes of 1/12/24

Announcements

- Office Hour and Study Session schedule will be published over the weekend. We are still confirming details.
- Pretest is closed. You'll get full credit if you participated at all.
- Make sure you read the interpretation guide which is available on our Canvas page.
- Thanks for participating in the survey. We look forward to reading it.
- Notes, annotated notes, and class recordings available on Canvas home page.
- No class Monday

- Major procedure in mathematics:

1. Start with an intuitive concept.
2. Make a precise definition.
3. Use the definition to derive properties (mostly formulas) of the newly defined concept.
4. Work with the properties, and go back to the definition only when necessary (and rarely in practice).

- In Math 1210 we will apply this procedure to the concepts of
  - **Limits**
  - **Continuity** (can draw graph without lifting the pencil)
  - **Derivatives** (location $\longrightarrow$ velocity)
  - **Integrals** (velocity $\longrightarrow$ location)

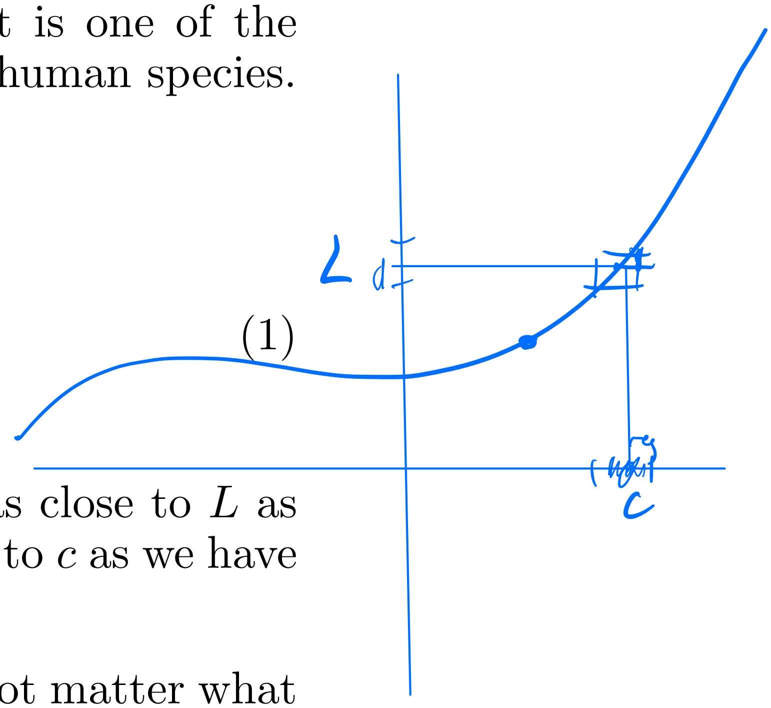
- We have already applied step 1 to all of these concepts.

1.2 Rigorous Study of Limits

- **Full Disclosure:** This is probably the most abstract class discussion of the whole 3 semester sequence. But it's worth it. The precise definition of the concept of limit is one of the great accomplishments of the human species.

- What does it mean that

$$\lim_{x \longrightarrow c} f(x) = L? \tag{1}$$



- Informal: we can make $f(x)$ as close to L as we wish by choosing x as close to c as we have to.

- We never pick $x = c$. It does not matter what $f(c)$ is or even if it is defined.

- Precise definition due to Karl Weierstrass (1815–1897).

- Think of it as a competition: I want to show that (1) is true, you want to prove me wrong.

- So you challenge me: How do you make $f(x)$ this close to L?

- I say: All I do is pick x that close to c.

- Since I have to show that I can meet any and all such challenges we cannot use specific numbers.

- We have to use variables. That's OK. We know algebra!
- The traditional choice of the variables in this context is ϵ (the lower case Greek letter epsilon) and δ (the lower case Greek letter delta).

- **Definition:** We say that

$$\lim_{x \longrightarrow c} f(x) = L$$

("*The limit of $f(x)$ as x approaches c equals L*") if for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - L| < \epsilon$$

for all x satisfying

$$0 < |x - c| < \delta.$$

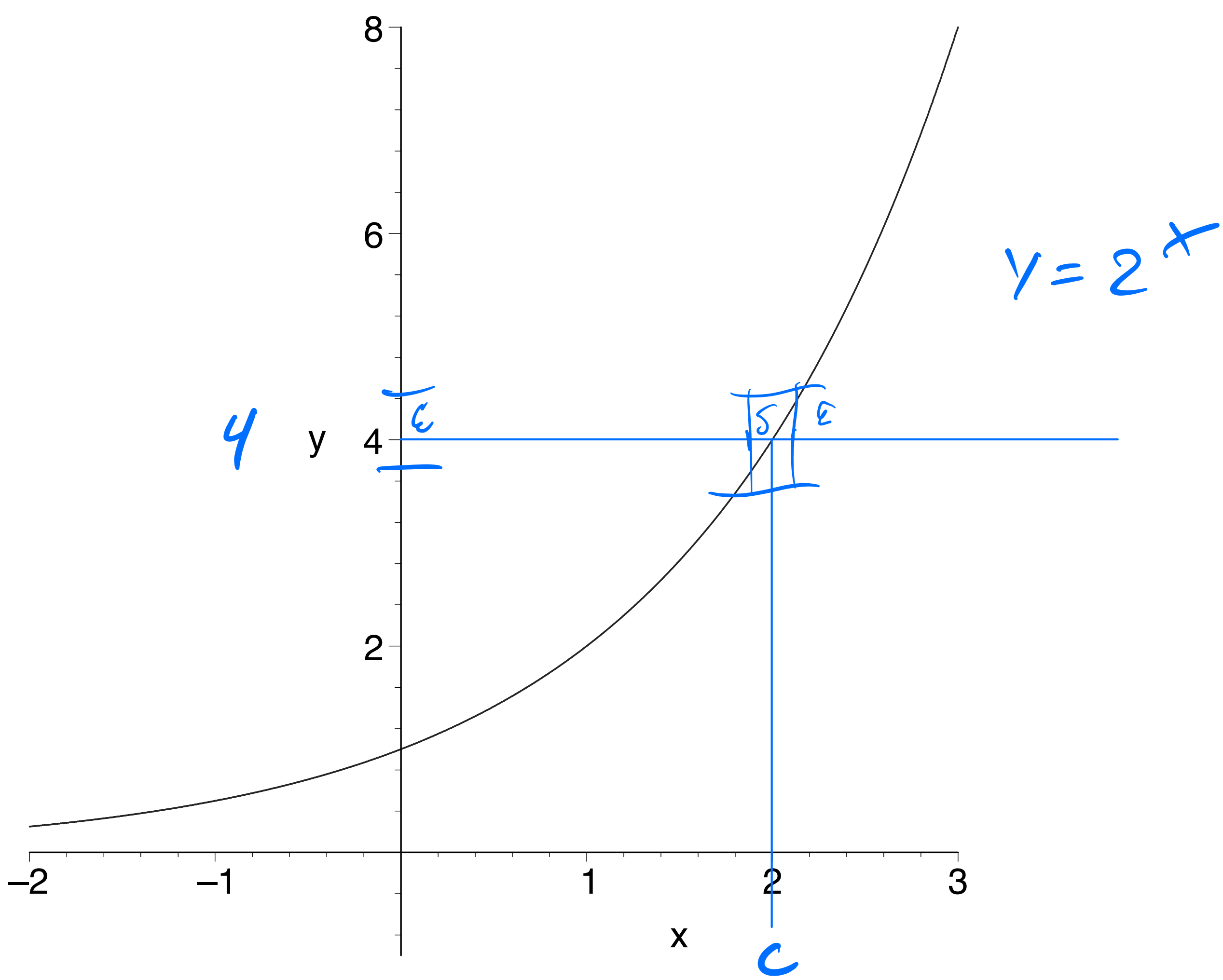


Figure 1. The (ϵ, δ) definition of the limit.

- We can also write the implication as

$$0 < |x - c| < \delta \qquad \Longrightarrow \qquad |f(x) - L| < \epsilon.$$

- Note that if δ works then any smaller value works as well.

- Example 2, page 63.

$$\lim_{x \longrightarrow 4} (3x - 7) = 5.$$

- (This is obvious. The point is to illustrate the definition, not that to convince you that $3x - 7$ approaches 5 as x approaches 4.)

- Compare with

$$\lim_{x \longrightarrow c} f(x) = L$$

- We have

$$c = 4 \qquad L = 5 \qquad f(x) = 3x-7$$

- We want

$$0 < |x - 4| < \delta \qquad \Longrightarrow \qquad |(3x - 7) - 5| < \epsilon$$

- Given ϵ, how do we pick δ?

$$\begin{aligned} |f(x) - L| &= |3x - 7 - 5| \\ &= |3x - 12| \\ &= 3|x-4| < \varepsilon \\ |x-4| &< \frac{\varepsilon}{3} = \delta \end{aligned}$$

- Example 3:

$$\lim_{x \longrightarrow 2} \frac{2x^2 - 3x - 2}{x - 2} = 5.$$

$$\frac{0}{0}$$

- This is closer to the kind of limit we actually want to compute. Both numerator and denominator go to zero.

$c = 2$

$L = 5$

$$\frac{2x^2 - 3x - 2}{x - 2} = \frac{(x-2)(2x+1)}{(x-2)} \qquad x \neq 2$$

$$= 2x + 1 \qquad \frac{ax}{a} = x$$

$$|2x+1-5| = |2x-4| = 2|x-2| < \varepsilon$$

$$|x-2| < \frac{\varepsilon}{2} = \delta$$

$$2(\;) < [\;]$$

$$(\;) < \frac{[\;]}{2}$$

$$\frac{3}{4} = \frac{6}{8} = \frac{2 \cdot 3}{2 \cdot 4}$$

- We don't have to have specific numbers.
- Example 5: Assume $c > 0$ and show that

$$\lim_{x \longrightarrow c} \sqrt{x} = \sqrt{c}.$$

$(a-b)(a+b)=a^2-b^2$

$$|\sqrt{x}-\sqrt{c}| < \frac{(\sqrt{x}-\sqrt{c})(\sqrt{x}+\sqrt{c})}{\sqrt{x}+\sqrt{c}}$$

$$= \left| \frac{x-c}{\sqrt{x}+\sqrt{c}} \right| < \varepsilon$$

$$|x-c| < \varepsilon(\sqrt{x}+\sqrt{c})$$

$$< 3\varepsilon\sqrt{c} = \delta$$

$x < 4c$

$\sqrt{x} < 2\sqrt{c}$

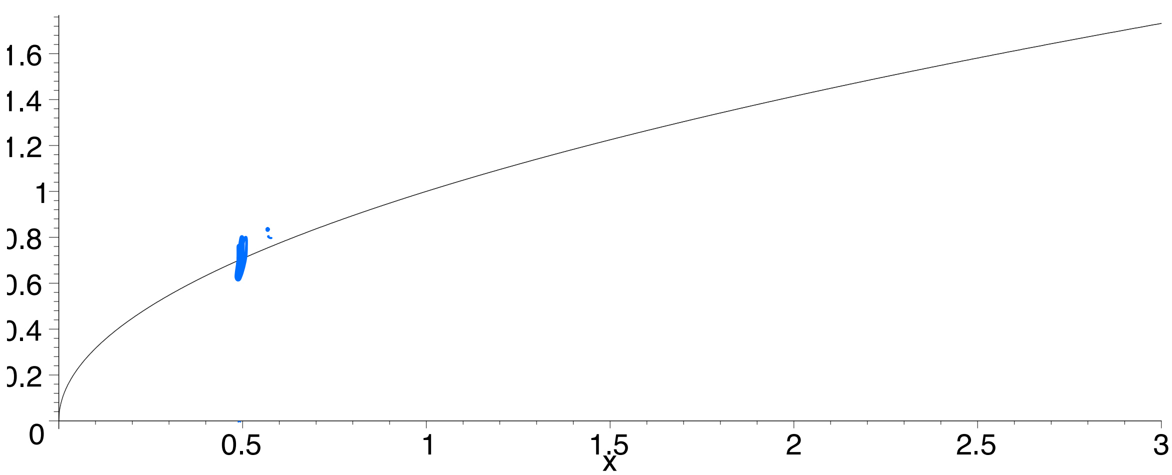


Figure 2. Square Root Function.

We defined in (ϵ, δ) language what we mean by

$$\lim_{x \longrightarrow c} f(x) = L$$

Can you say in (ϵ, δ) language that

$$\lim_{x \longrightarrow c} f(x) \neq L?$$

- This is problem 32 on page 67 of the textbook, and also problem 17 in hw 2.

- Applying the $\epsilon - \delta$ definition can be tricky, and is only a last resort.
- Usually we apply appropriate properties of limits.
- So what are those properties?

Main Limit Theorem

- See textbook, page 68.
- Let n be a positive integer, k a constant, and f and g functions that have limits at c. Then:

1. $\lim\limits_{x \longrightarrow c} k = k.$

2. $\lim\limits_{x \longrightarrow c} x = c.$

3. $\lim\limits_{x \longrightarrow c} kf(x) = k \lim\limits_{x \longrightarrow c} f(x).$

4. $\lim\limits_{x \longrightarrow c} \big(f(x) + g(x)\big) = \lim\limits_{x \longrightarrow c} f(x) + \lim\limits_{x \longrightarrow c} g(x).$

5. $\lim\limits_{x \longrightarrow c} \big(f(x) - g(x)\big) = \lim\limits_{x \longrightarrow c} f(x) - \lim\limits_{x \longrightarrow c} g(x).$

6. $\lim\limits_{x \longrightarrow c} \big(f(x) \cdot g(x)\big) = \Big(\lim\limits_{x \longrightarrow c} f(x)\Big) \cdot \Big(\lim\limits_{x \longrightarrow c} g(x)\Big).$

7. $\lim\limits_{x \longrightarrow c} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \longrightarrow c} f(x)}{\lim\limits_{x \longrightarrow c} g(x)} \quad \text{if} \quad \lim\limits_{x \longrightarrow c} g(x) \neq 0.$

8. $\lim\limits_{x \longrightarrow c} \big(f(x)\big)^n = \Big(\lim\limits_{x \longrightarrow c} f(x)\Big)^n.$

9. $\lim\limits_{x \longrightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \longrightarrow c} f(x)} \quad \text{if} \quad f(x) \geq 0$ when n is even.

- The main limit theorem let's us compute many limits right away:
- Example 1:

$$\lim_{x \longrightarrow 3} 2x^4 = 2 \lim_{x \longrightarrow 3} x^4 = 2 \left(\lim_{x \longrightarrow 3} x \right)^4 = 2 \times 3^4 = 2 \times 81 = 162.$$

- in fact, for any polynomial p we have

$$\lim_{x \longrightarrow c} p(x) = p(c).$$

(very soon we will state this fact by simply saying that polynomials are continuous).

- It would be tedious to show all nine points of the Main Limit Theorem via the $\epsilon - \delta$ definition. If time allows let's just do point 4.

$$\lim_{x \to c} f(x) = F$$

$$\lim_{x \to c} g(x) = G$$

$$1 = |3-4| \le |3| + |4| = 7$$

$$|a+b| \le |a| + |b|$$

$$\lim \left(f(x) + g(x) \right) = F + G$$

$$\left| f(x) + g(x) - (F + G) \right|$$

$$= \left| f(x) - F + g(x) - G \right|$$

$$< \left| f(x) - F \right| + \left| g(x) - G \right|$$

$$< \frac{\varepsilon}{2} \qquad\qquad < \frac{\varepsilon}{2}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

pick that δ

$$\lim_{x \to c} \left(f(x) + g(x) \right) = F + G = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$$