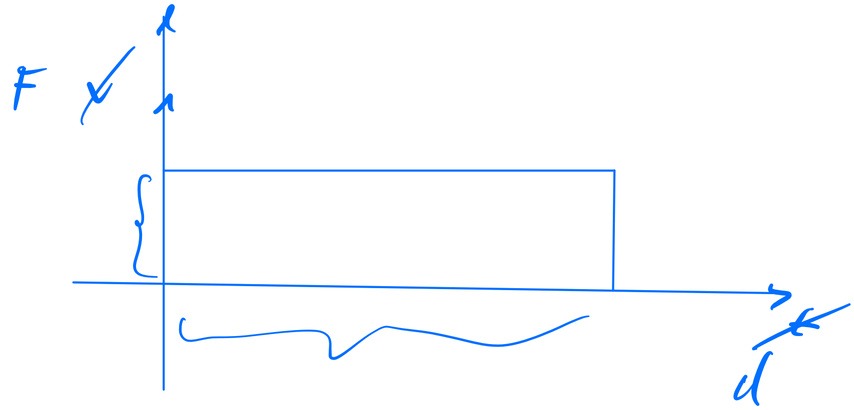
- no recording yesterday
- Notes today: wrong link

# Math 1210-23

# Notes of 4/10/24




## 5.5 Work

- Assuming that the force is constant,

$$\textbf{work} = \textbf{force} \times \textbf{distance}$$

- Example: Lifting 10lbs 1 foot requires the same amount of work (10 foot-pounds) as lifting 1 pound 10 feet.
- What if the force is not constant?
- Why, then work is the integral of force with respect to distance!
- We've seen this before.
- If velocity is constant than distance equals velocity times time. If not we get the integral of velocity with respect to time.
- If the height is constant then area equals width times height. If not we integrate the height.
- If the area of the cross-section is constant then the volume equals length times that area. Otherwise we integrate the area of the cross-section.

- We could get the same insight using the textbook technique:

1. Slice
2. Approximate
3. Sum
4. Take the limit
5. Recognize the definite integral
6. Integrate

- But with our experience this should no longer be necessary. Just think in terms of the graph of force versus distance, and Riemann sums. It's all the same as before.

- We can, and sometimes will, measure work in foot-pounds. However, a more scientific unit is

$$1 \text{ Joule} = 1 \underbrace{\text{Newton}}_{\text{Force}} \underbrace{\text{Meter}}_{\text{Length}}$$

where

$$1 \text{ meter} \approx 3.28 \text{ feet}$$

$$1 \text{ Newton} \approx 0.25 \text{ pounds}$$

## Springs

- Elastic springs provide a simple example for a non-constant force.
- Every spring has a natural length $L$. Let $x$ denote the departure of the spring's length from $L$. If $x$ is negative we compress the spring, otherwise we expand the spring.
- The force caused by expansion or compression is modeled by **Hooke's Law**:

  $$F = kx$$

  where the **spring constant** $k$ is positive.
- The work required to expand a spring from length $L$ to $L + E$ ($E$ for "extension") is

  $$W = \int_0^E kx\mathrm{d}x = \int_L^{L+E} k(x - L)\mathrm{d}x = \frac{kE^2}{2}.$$

- Note that $E$ could be negative (in which case we compress the spring).

- Example 1, textbook. A spring has a natural length of 0.2m and it takes a force of 12 Newtons to keep it extended by 0.04m. Find the work required to extend the spring from its natural length to a length of 0.3m.

$$F = kx$$

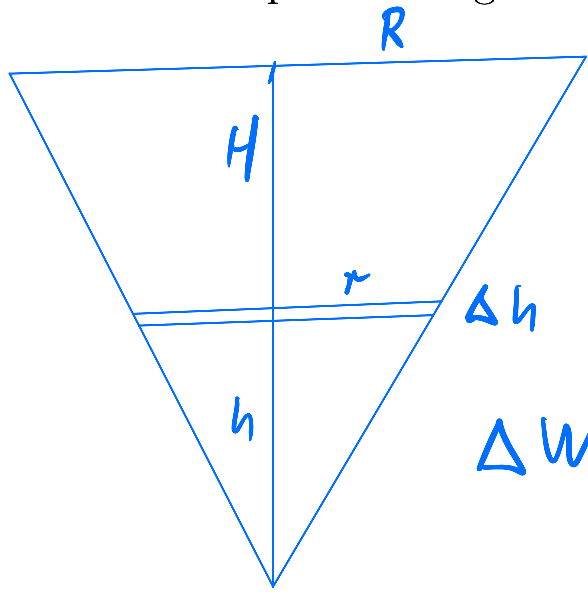
$$12 = k \cdot 0.04$$

$$k = \frac{12}{0.04} \simeq 300$$

$$W = \int_0^{0.1} 300x\,dx = 300\left[\frac{x^2}{2}\right]_0^{0.1}$$

$$= 300 \cdot \frac{0.01}{2} = 1.5 \text{ Joules}$$

- Example 2, page 303, textbook, modified. A tank in the shape of an inverted right circular cone of radius $R$ and height $H$ is full of water. Find the work required to pump out the water (by lifting it to the top of the cone).
- Let $\delta$ denote the specific weight of water.




$$\Delta W = \delta \pi r^2 \Delta h \, (H-h)$$

$$W = \int_0^H \delta \pi r^2 (H-h)\, dh$$

$$\frac{r}{h} = \frac{R}{H}$$

$$r = \frac{R}{H} h$$

$$W = \int_0^H \delta \pi \frac{R^2 h^2}{H^2} (H-h)\, dh$$

$$= \delta \pi \frac{R^2}{H^2} \int_0^H h^2 (H - h)\, dh$$

$$= \delta \pi \frac{R^2}{H^2} \int_0^H h^2 H - h^3 \, dh$$

$$= \delta \pi \frac{R^2}{H^2} \left[ \frac{Hh^3}{3} - \frac{h^4}{4} \right]_0^H$$

$$= \delta \pi \frac{R^2 H^4}{H^2} \left[ \frac{1}{3} - \frac{1}{4} \right]$$

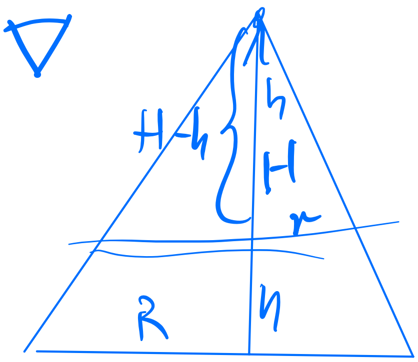
$$W = \frac{1}{12} \cdot \delta \pi R^2 H^2$$

$$lb \cdot ft \quad \Longleftrightarrow \quad \frac{lb}{ft^3} \cdot ft^2 \cdot ft^2 = lb \cdot ft$$

$$\delta \quad \frac{lb}{ft^3}$$

- What if the cone is not inverted? Expectations?

$\nabla$




$$W = \delta\pi \int_0^H r^2 (H-h)\, dh$$

$$= \delta\pi \frac{R^2}{H^2} \int_0^H (H-h)^3\, dh$$

$$= \delta\pi \frac{R^2}{H^2} \left[ -\frac{(H-h)^4}{4} \right]_0^H$$

$$= \delta\pi \frac{R^2}{H^2} H^4 \cdot \frac{1}{4}$$

$$= \frac{1}{4} \delta\pi R^2 H^2$$

$$= 3\, W_{\nabla}$$

$$\frac{r}{H-h} = \frac{R}{H}$$

$$r = \frac{R}{H}(H-h)$$

## Escape Velocity, again

- Here is another intriguing calculation.
- Recall that we saw earlier that earth has a certain escape velocity

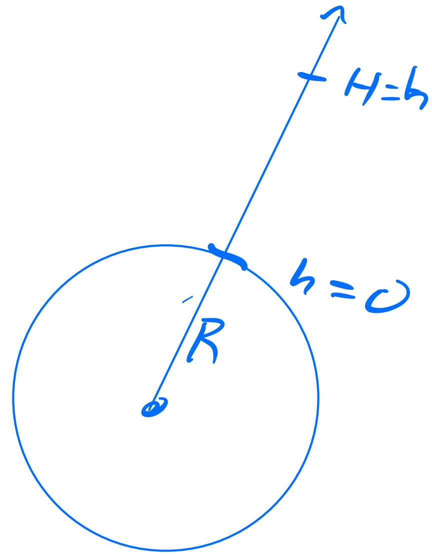
$$v = \sqrt{2gR} \approx 6.93 \text{ miles per second}$$

  where $R$ is the radius of earth, and $g$ is gravity on the surface of the earth. We saw this by analyzing a differential equation. See Example 5 on page 207 of the textbook, or the notes of March 21.

- Here is a work based approach to the same problem (naturally giving the same answer).
- How much work is required to lift an object from sea level to a height $H$? Let $w$ denote the weight of the object a sea level, $R$ the radius of the earth, $h$ the height above sea level, and $H$ the final height.
- $H$ could be infinity. It turns out that the work required to lift an object to infinity is finite! In fact we will compute the escape velocity by equating that work to kinetic energy and solving for velocity.

surface weight $= mg$

$$\text{weight}(h) = \frac{mgR^2}{(R+h)^2}$$




$$W = \int_0^H \frac{mgR^2}{(R+h)^2}\,dh$$

$$= mgR^2 \int_0^H \frac{1}{(R+h)^2}\,dh$$

$$= mgR^2 \left[\frac{-1}{R+h}\right]_0^H$$

$$= mgR^2 \left[-\frac{1}{R+H} + \frac{1}{R}\right]$$

$H \to \infty$ $\quad \to mgR^2 \cdot \frac{1}{R} = \cancel{m}gR = \frac{1}{2}\cancel{m}v^2$

$$v^2 = 2gR$$

$$v = \sqrt{2gR}$$