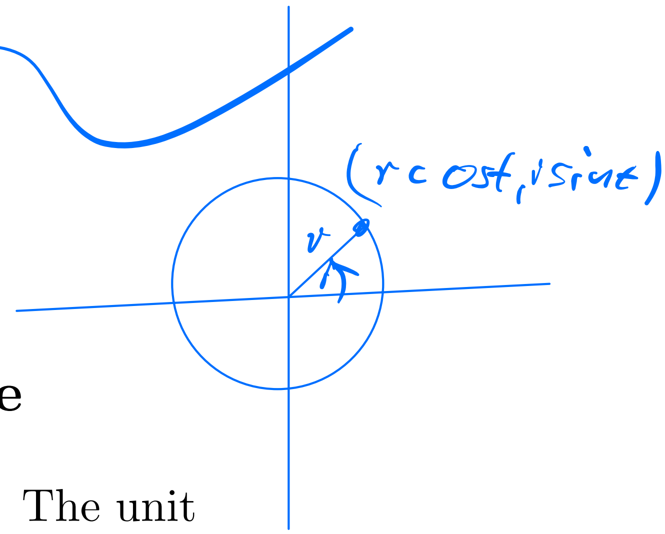
Math 1210-23

Notes of 4/9/24



5.4 Length of a Plane Curve

- Let's start with a familiar example. The unit circle can be described as

$$x = \cos t \quad \text{and} \quad y = \sin t$$

- We think of t as the angle the segment from the origin to (x, y) makes with the x-axis, but we can also think of t as time, for example.
- In general a **parametric curve** is described by

$$x = f(t), \qquad y = g(t), \qquad a \le t \le b$$

- We will assume that f and g are differentiable.
- Note that parametric curves include ordinary functions:

$$x = t \quad \text{and} \quad y = g(t) = g(x)$$

- Let's illustrate with a couple more examples:
- Example 1:

$$x = 2t + 1, \qquad y = t^2 - 1, \qquad 0 \le t \le 3$$

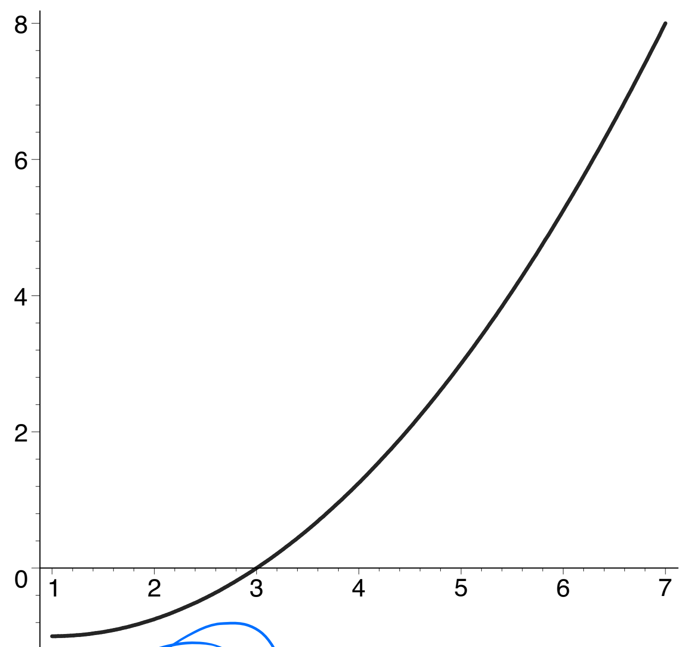


Figure 1. $x = 2t + 1$, $y = t^2 - 1$, $0 \le t \le 3$.

- In this case, we can eliminate t and get a function $y(x)$.

$2t = x + 1$

$t = \frac{x+1}{2}$

$y = t^2 - 1$

$= \left(\frac{x+1}{2}\right)^2 - 1$

Example 2: The Cycloid

- Interesting example with lots of fascinating properties!

$$x = t - \sin t \quad \text{and} \quad y = 1 - \cos t$$

- It's the trajectory of a point on the rim of a wheel (rolling to the right in the Figure.)

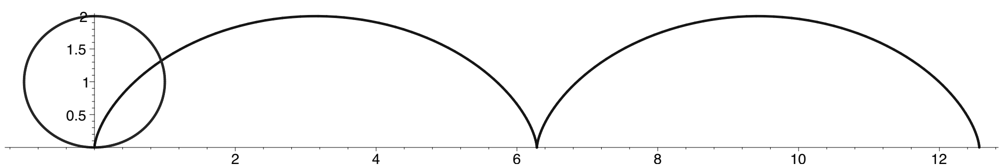
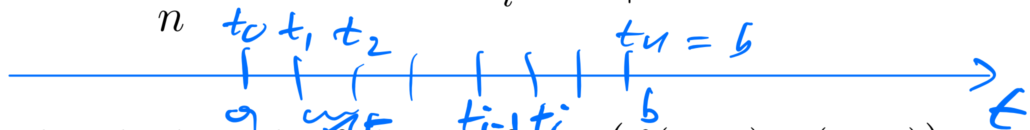


Figure 2. A Cycloid.

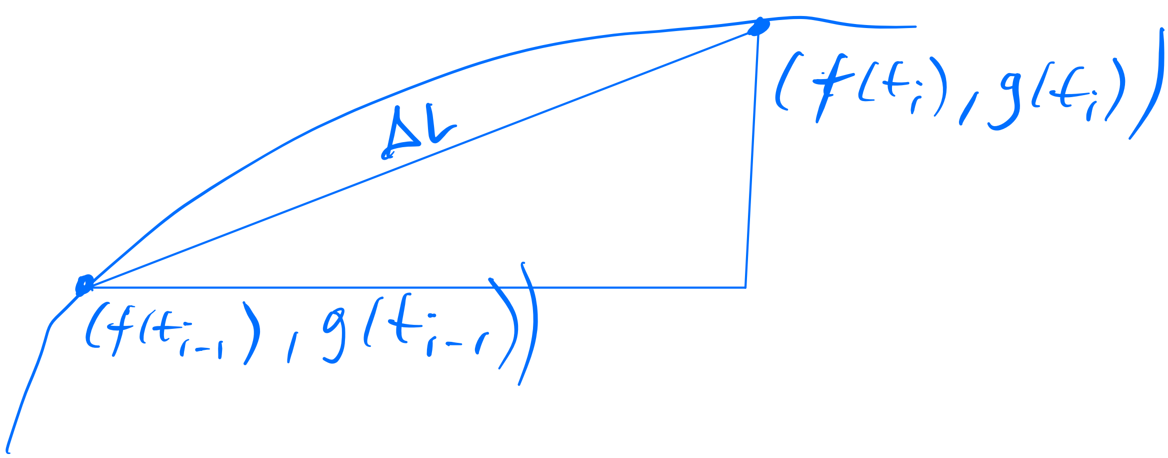
Arc Length

- Today's main topic: How to compute the length of such a curve.
- Idea: Partition the interval $[a, b]$ into subintervals, approximate on the subintervals, let the number of subintervals go to infinity and their widths go to zero, take the limit, recognize a Riemann Sum, interpret it as an integral ...just as we did for volumes.
- So we let

$$\Delta t = \frac{b-a}{n} \qquad \text{and} \qquad t_i = a + i\Delta t$$



- Consider the length of the arc from $\big(f(t_{i-1}), g(t_{i-1})\big)$ and $\big(f(t_i), g(t_i)\big)$.



- It's approximately equal to the length ΔL of the line segment between those two points which is given by

$$\Delta L = \sqrt{\big(f(t_i) - f(t_{i-1})\big)^2 + \big(g(t_i) - g(t_{i-1})\big)^2}$$

- Noting that

$$\begin{aligned} f(t_i) - f(t_{i-1}) &\approx f'(t_i)\Delta t \quad \text{and} \\ g(t_i) - g(t_{i-1}) &\approx g'(t_i)\Delta t \end{aligned}$$

we get

$$\begin{aligned} \Delta L &\approx \sqrt{\big(f'(t_i)\Delta t\big)^2 + \big(g'(t_i)\Delta t\big)^2} \\ &= \sqrt{\big(f'(t_i)\big)^2 + \big(g'(t_i)\big)^2}\Delta t \end{aligned}$$

- So the length of the whole curve is

$$\begin{aligned} Ł &= \lim_{n \longrightarrow \infty} \sum_{i=1}^{n} \sqrt{\big(f'(t_i)^2\big) + \big(g'(t_i)^2\big)}\Delta t \\ &= \int_a^b \sqrt{f'^2(t) + g'^2(t)}\mathrm{d}t. \end{aligned}$$

- This is something we can compute!

$$f(t) =$$

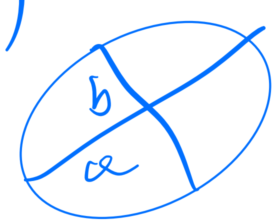
- Example 3: Circle around origin with radius r, say. The circumference of that circle is $2\pi r$

$$L = \int_0^{2\pi} \left(\left((r\cos t)'\right)^2 + \left((r\sin t)'\right)^2 \right)^{1/2} dt$$

$$= \int_0^{2\pi} \sqrt{r^2 \sin^2 t + r^2 \cos^2 t}\, dt$$

$$= \int_0^{2\pi} r\, dt = 2\pi r$$

$$\int_a^b c\, dt = ct\Big|_a^b = c(b-a)$$

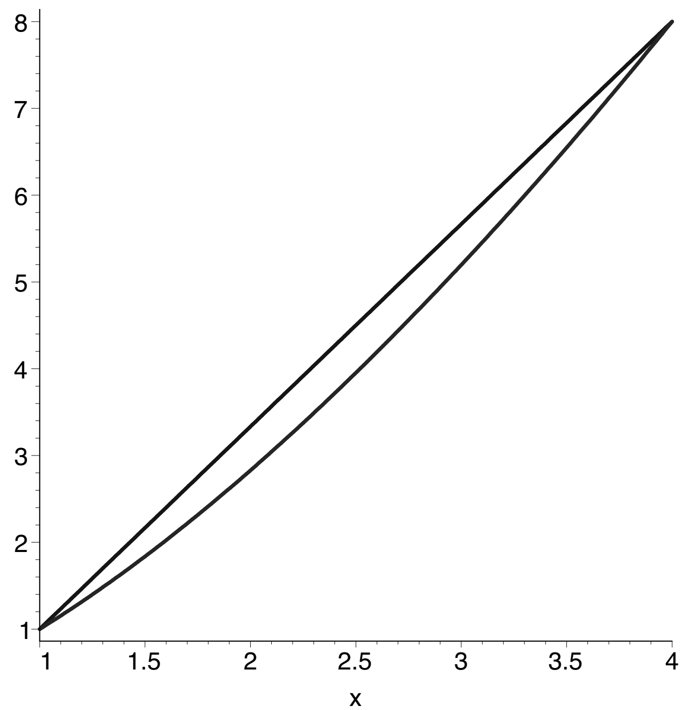


$f(x) = x^{3/2}$

- Example 5:

$$x = t, \qquad y = t^{\frac{3}{2}}, \qquad 1 \leq t \leq 4$$

(4, 8)



(1, 1)

Figure 3. Example 5.

- That arc should have length very close to the distance between the points $(1, 1)$ and $(4, 8)$ which is

$$\sqrt{(4-1)^2 + (8-1)^2} = \sqrt{58} \approx 7.55$$

- Carrying out the integration we get:

$$L = \int_1^4 \sqrt{1 + \left(\frac{d}{dt} t^{3/2}\right)^2}\, dt$$

$$= \int_1^4 \sqrt{1 + \left(\frac{3}{2} t^{1/2}\right)^2}\, dt$$

$$= \int_1^4 \left(1 + \frac{9}{4} t\right)^{1/2} dt$$

$$= \frac{4}{9}\frac{2}{3}\left(1 + \frac{9}{4} t\right)^{3/2} \Big|_1^4$$

$$= \frac{8}{27}\left(10^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right)$$

- Numerically,

$$L = 7.63 \approx 7.55$$

- In general, we have a curve of the form $y = f(x)$ we can use $x = t$ and $y = f(t)$ to get

$$\begin{aligned} L &= \int_a^b \sqrt{1 + f'^2(t)}\mathrm{d}t \\ &= \int_a^b \sqrt{1 + f'^2(x)}\mathrm{d}x \end{aligned}$$

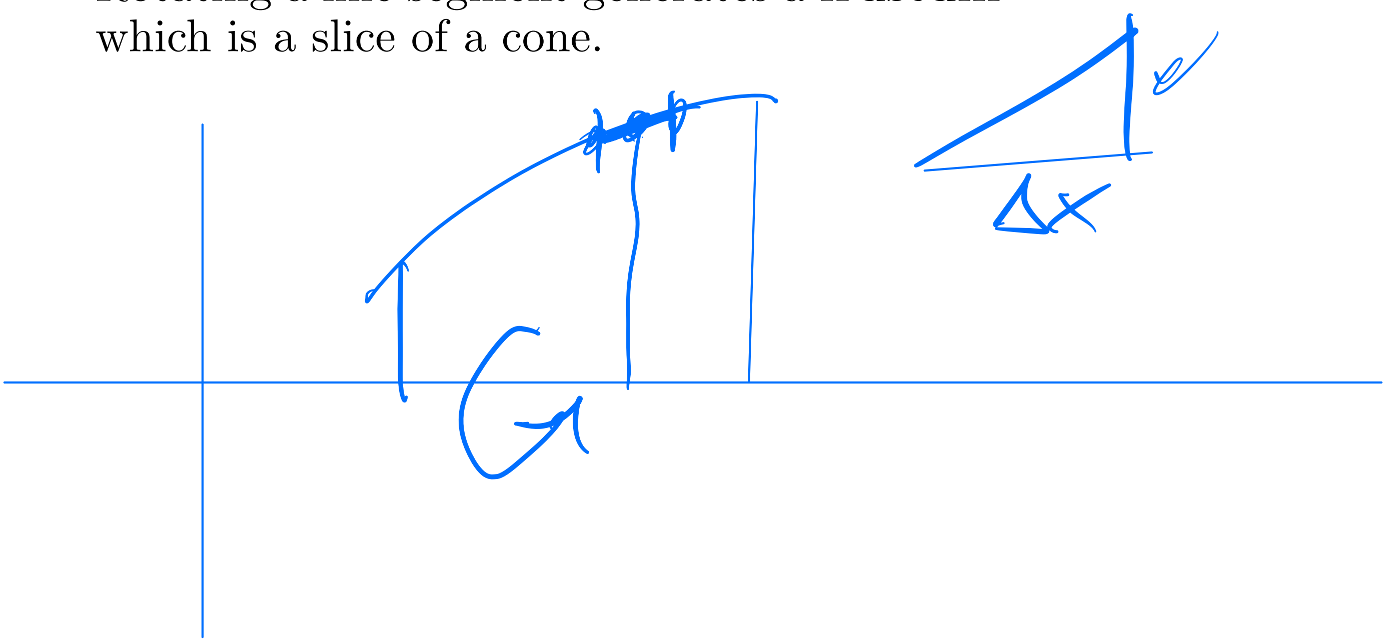
$x = t \qquad y = f(t) = f(x)$

(y')

$$L = \int_a^b \sqrt{1 + (f'(t))^2}\, dt$$

Surface Area

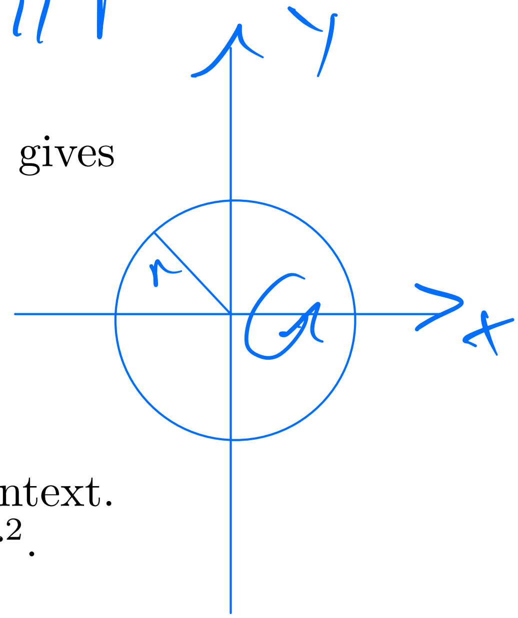
- We can expand these concepts to computing the surface area of a solid of revolution.
- Suppose we rotate the graph of $y = f(x)$ around the x-axis.
- Rotating a line segment generates a **frustum** which is a slice of a cone.



The surface area of the inclined part of the frustum is approximately

$$\begin{aligned}\Delta A &= 2\pi\left(\frac{y_{i-1}+y_i}{2}\right)\sqrt{\Delta x^2 + \left(f'(x)\Delta x\right)^2}\\ &= \pi(y_{i-1}+y_i)\sqrt{1+f'^2(x_i)}\,\Delta x\end{aligned}$$

sphere $\quad S = 4\pi r^2$



- Adding, taking the limit, ..., as usual, gives the integral

$$A = \int_a^b 2\pi f(x)\sqrt{1 + f'^2(x)}\mathrm{d}x$$

- Again, let's try this out in a familiar context. The surface area of a sphere equals $4\pi r^2$.

- We get:

$$x^2 + y^2 = r^2$$

$$y = f(x) = \sqrt{r^2 - x^2}$$

$$= (r^2 - x^2)^{1/2}$$

$$f'(x) = \frac{1}{2}(-2x)(r^2 - x^2)^{-1/2}$$

$$= -x(r^2 - x^2)^{-1/2}$$

$$f'(x)^2 = x^2(r^2 - x^2)^{-1}$$

$$A = \int_{-r}^{r} 2\pi \left(r^2 - x^2\right)^{1/2} \left(1 + x^2 \left(r^2 - x^2\right)^{-1}\right)^{1/2} dx$$

$$= 2\pi \int_{-r}^{r} \left(\left(r^2 - x^2\right)\left(1 + x^2 \left(r^2 - x^2\right)^{-1}\right)\right)^{1/2} dx$$

$$= 2\pi \int_{-r}^{r} \left(\left(r^2 - \cancel{x^2}\right) + \cancel{x^2}\right)^{1/2} dx$$

$$= 2\pi \int_{-r}^{r} r \, dx$$

$$= 4\pi r^2$$

---

sphere r

$$V = \frac{4}{3} \pi r^3$$

$$S = 4\pi r^2$$

Work