Math 1210-23

Notes of 4/8/24

More Volumes

- Example 6, page 286. The base of a solid is the region between one arc of $y = \sin x$ and the cross section perpendicular to the x-axis and parallel to the y-axis is an equilateral triangle. Find the volume of that solid.
- See Figure 14 on page 286 of the textbook.

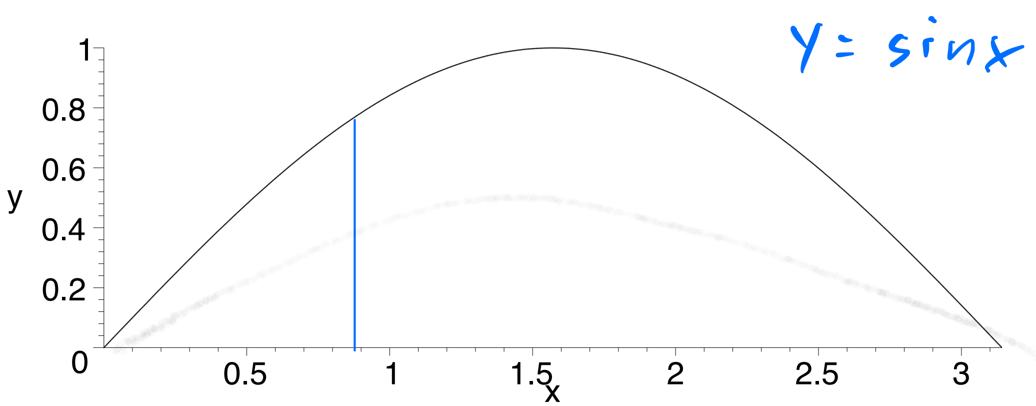
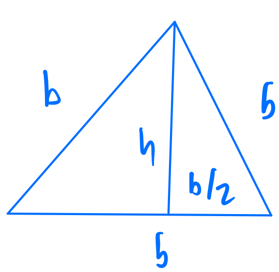


Figure 1. Base of Solid.



$$h^2 + \frac{b^2}{4} = b^2$$

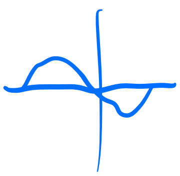
$$h^2 = b^2 - \frac{b^2}{4} = \frac{3}{4} b^2$$

$$h = \frac{\sqrt{3}}{2} b$$

$$A_\Delta = \frac{1}{2} b h = \frac{\sqrt{3}}{4} b^2 = \frac{\sqrt{3}}{4} \sin^2 x$$

$$V = \int_0^\pi \frac{\sqrt{3}}{4} \sin^2 x \, dx = \frac{\sqrt{3}}{4} \underbrace{\int_0^\pi \sin^2 x \, dx}_{= \pi/2}$$

$$= \frac{\sqrt{3}}{8} \pi$$



$$\int_0^{2\pi} \sin^2 x \, dx = \frac{1}{2} \int_0^{2\pi} \cos^2 x + \sin^2 x \, dx = \pi$$

- Tour de Force, Example 4, page 291. “Putting it All Together”.
- Let R be the region bounded by the y axis, the curve $y = 3 + 2x - x^2$, and the x-axis. Compute the volume of the solid obtained by rotating this region around

a) the x-axis

b) The y-axis

c) the line $y = -1$

d) the line $x = 4$.

- We will skip the actual computation of the integrals, and just set up the integrals.

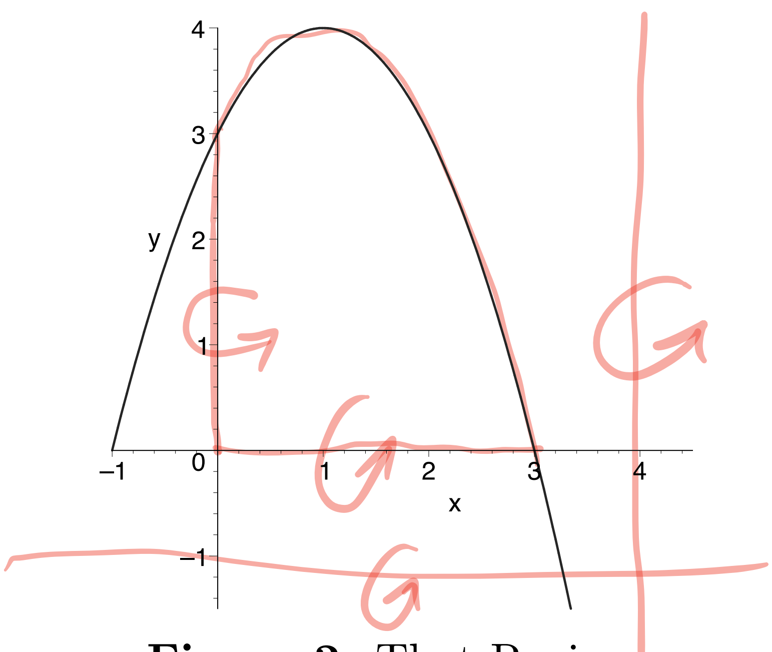


Figure 2. That Region.

- Note that in all 4 cases we will be integrating with respect to x running from 0 to 3.

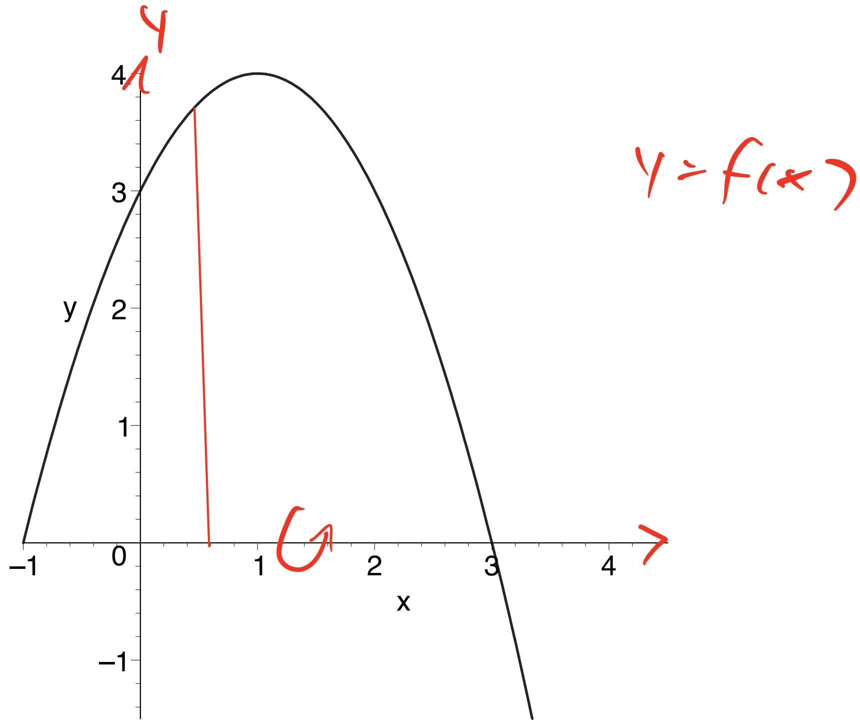


Figure 3. Rotating around the x-axis.

$$
\begin{aligned}
V &= \int_0^3 \pi y^2 \mathrm{d}x \\
&= \int_0^3 \pi \left(3 + 2x - x^2\right)^2 \mathrm{d}x \\
&= \frac{153\pi}{5}
\end{aligned}
$$

- This is an example of the method of disks.

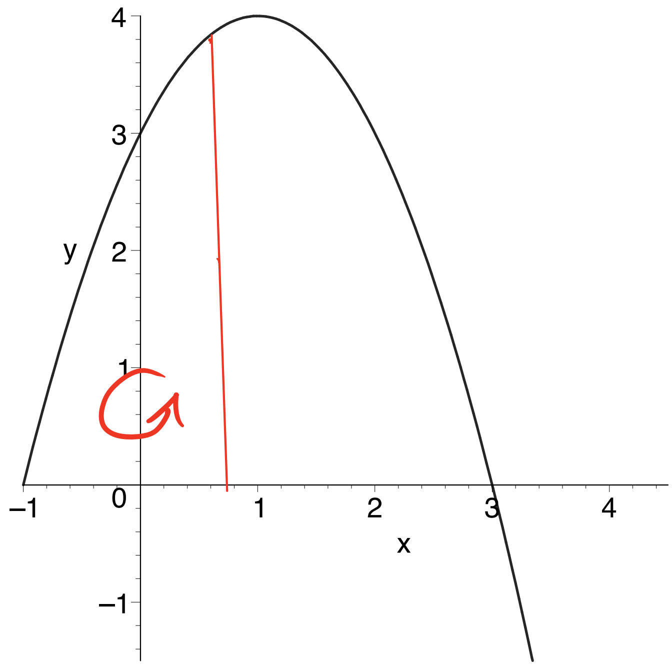


Figure 4. Rotating around the y-axis.

$$
\begin{aligned}
V &= \int_0^3 y \times 2\pi x \mathrm{d}x \\
&= \int_0^3 2\pi x(3 + 2x - x^2)\mathrm{d}x \\
&= \frac{45\pi}{2}
\end{aligned}
$$

- This is an example of the method of shells.

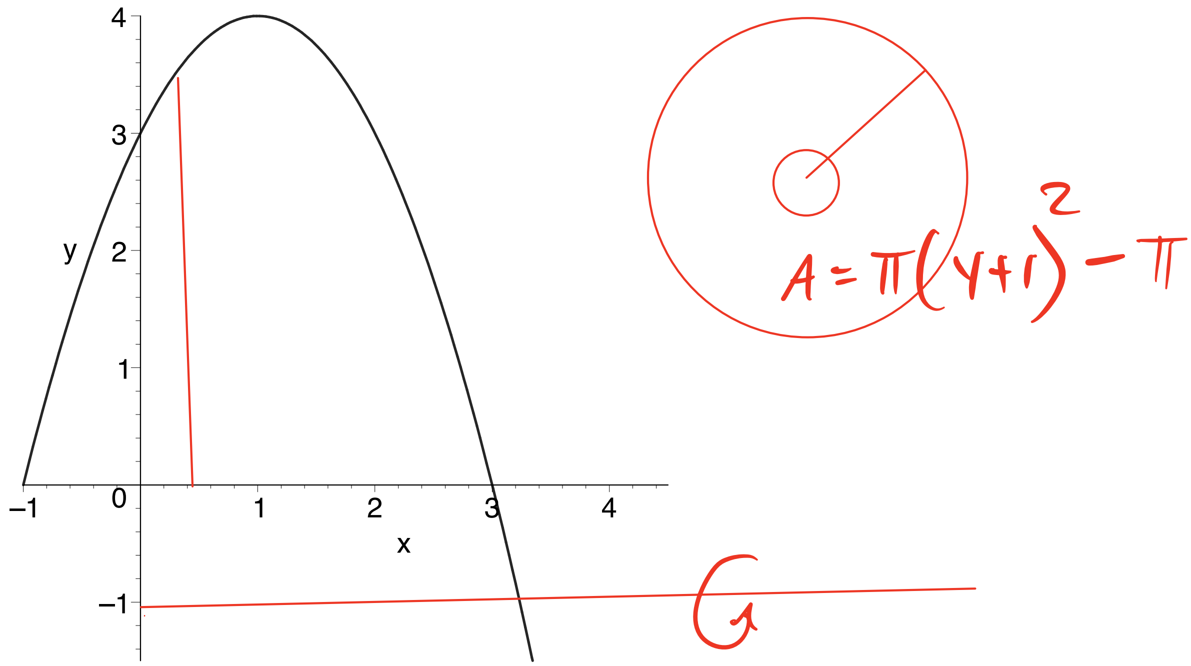


Figure 5. Rotating around the line $y = -1$.

$$
\begin{aligned}
V &= \int_0^3 \pi\Big((y+1)^2 - 1^2\Big)\mathrm{d}x \\
&= \pi\int_0^3 (4+2x-x^2)^2 - 1\mathrm{d}x \\
&= \frac{243\pi}{5}
\end{aligned}
$$

- The textbook calls this the method of washers.

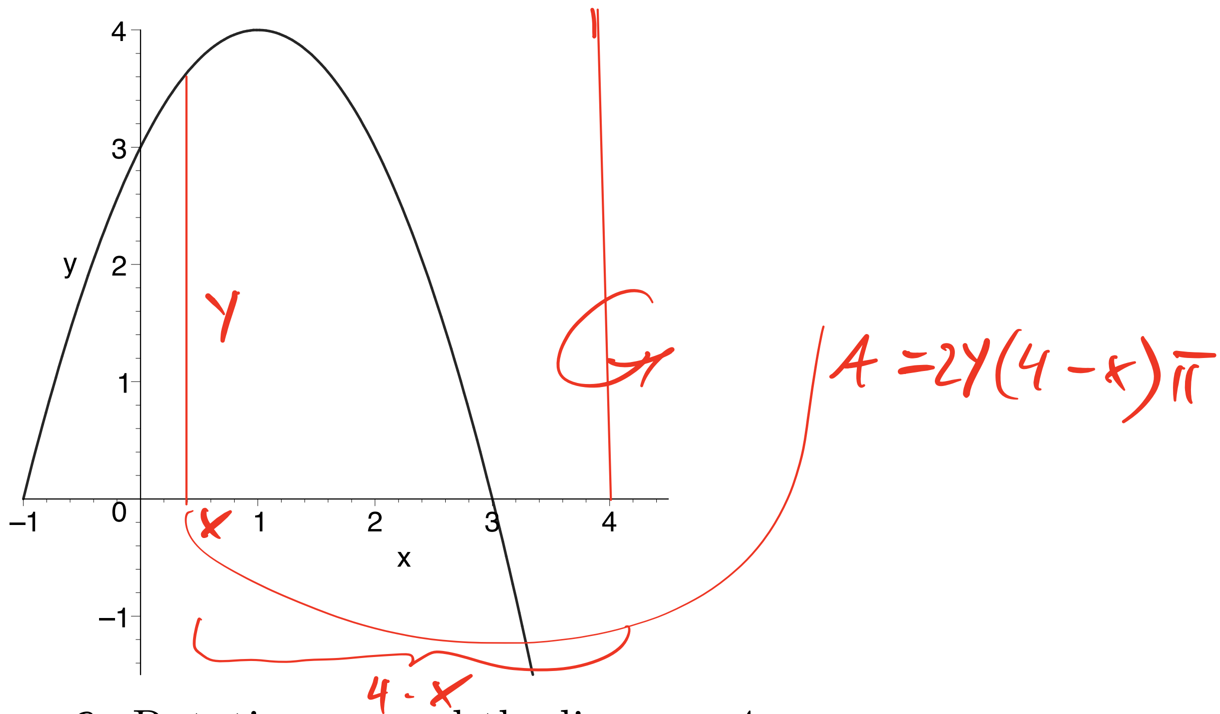


Figure 6. Rotating around the line $x = 4$.

$$
\begin{aligned}
V &= \int_0^3 2\pi(4-x)y\,\mathrm{d}x \\
&= \int_0^3 2\pi(4-x)(3+2x-x^2)\,\mathrm{d}x \\
&= \frac{99\pi}{2}
\end{aligned}
$$

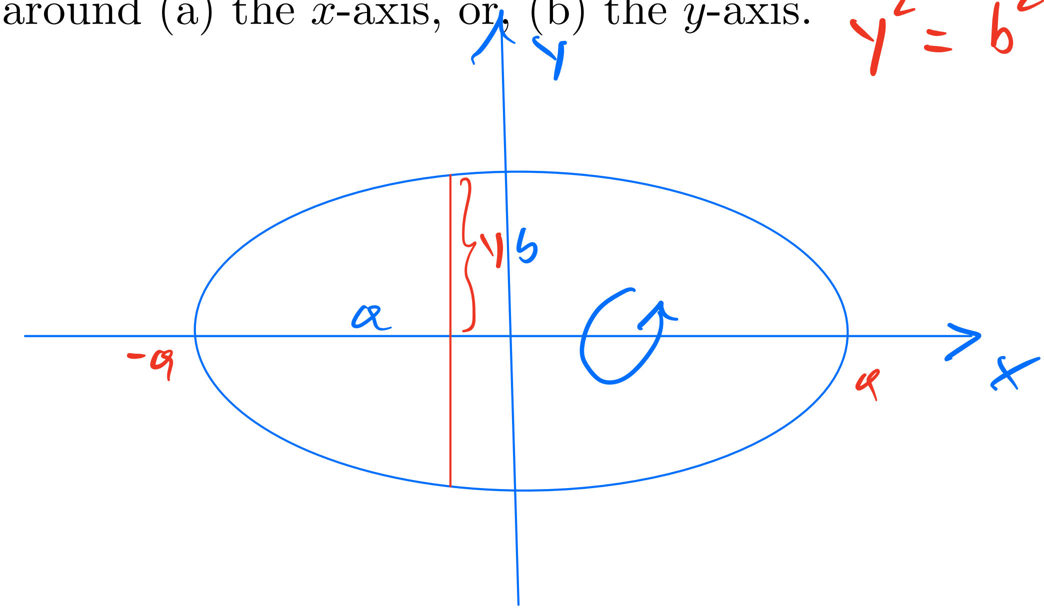
- This is another example of the method of shells.

$a = b = r \quad V = \frac{4}{3}\pi r^3$

Example: Compute the volume of the solid obtained by rotating the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

around (a) the x-axis, or, (b) the y-axis. $\quad y^2 = b^2 - \frac{b^2x^2}{a^2}$



$$V = \int_{-a}^{a} \pi y^2 \, dx = \int_{-a}^{a} \pi \left(b^2 - \frac{b^2x^2}{a^2} \right) dx$$

$$= b^2 \int_{-a}^{a} \pi \left(1 - \frac{x^2}{a^2} \right) dx$$

$$= b^2 \pi \left[x - \frac{x^3}{3a^2} \right]_{-a}^{a}$$

$$= b^2 \pi 2 \left[a - \frac{a^3}{3a^2} \right] = 2\pi b^2 \cdot \frac{2}{3} a \qquad x\text{-axis}$$

$$= \frac{4}{3}\pi b^2 a$$

y-axis

$V = \frac{4}{3}\pi a^2 b$

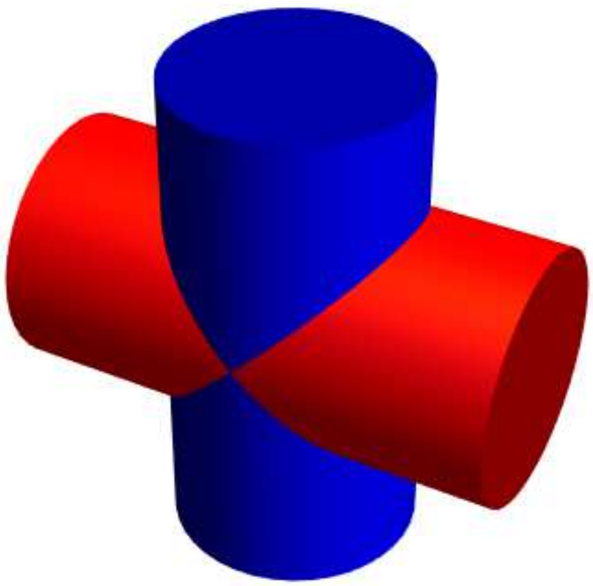
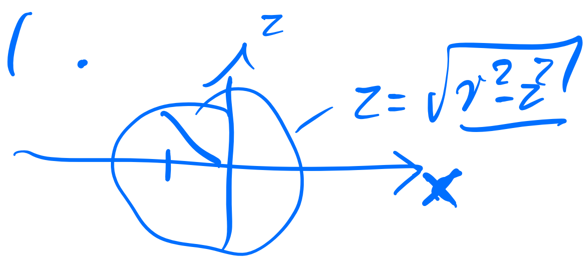


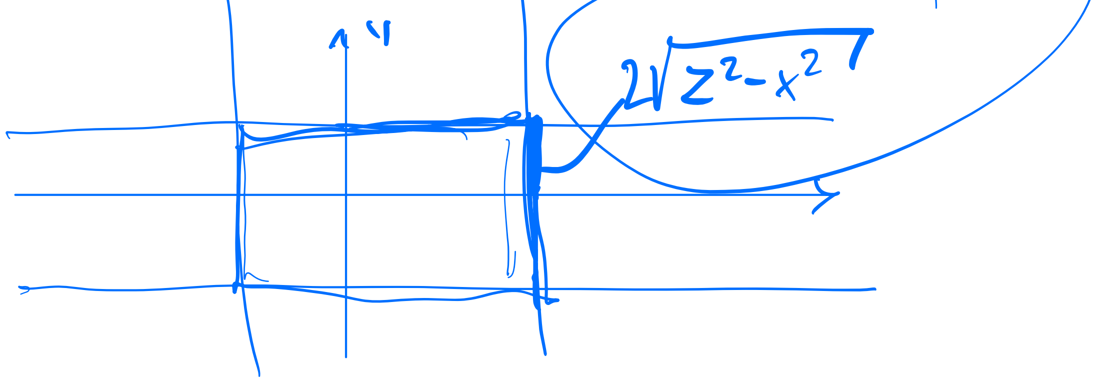
Figure 7. Intersecting Pipes.

Example: Problem 29, page 287. Find the volume of the "+" sign formed by two intersecting cylinders, both of length L and radius r, as shown in Figure[1] 7.

V : intersection vol.



[1] from https://i.stack.imgur.com/IUAHL.png, see also Figure 16 on page 287 of the textbook



$$A\,\square = 4\pi(r^2 - z^2)$$

$$V_{\text{inters.}} = \int_{-r}^{r} 4\pi(r^2 - z^2)\,dz$$

$$= 4\pi\left[r^2 z - \frac{z^3}{3}\right]_{-r}^{r}$$

$$= 4\pi\underbrace{\left[r^3 - \frac{r^3}{3}\right]}_{\frac{2}{3}r^3}2$$

$$= \frac{16}{3}\pi$$

- One more example: You drill a hole of radius r through a sphere of radius R such that the axis of the hole passes through the center of the sphere. Compute the volume of the Ring that's left, and the volume of the part that has been removed.

exercise