• For your info, here is a transcript of the exam instructions.

• Notice in particular items 7 and 8.

Instructions

1. This exam is closed books and notes, no electronics, and no scratch paper.

2. Use these sheets to record your work and your results. Use the space provided, and the back of these pages if necessary. **Show all work.** Unless it’s obvious, indicate the problem each piece of work corresponds to, and for each problem indicate where to find the corresponding work.

3. Please note: **To avoid disruption and distraction I won’t be able to answer questions during the exam.** If you believe there is a mistake in one of the problems write down an appropriate note and if you are right you will receive generous credit.

4. Simplify any algebraic expressions and reduce any fractions. For indefinite integrals, be sure to list the integration constant when appropriate.
5. If you are done before the allotted time is up I recommend strongly that you stay and use the remaining time to check your answers.

6. All questions have equal weight.

7. Make sure you list an integration constant where appropriate.

8. When appropriate please list your final answer in the space provided after the equals sign in the problem statements.

\[ F(x) = \int x^2 + 1 \, dx = \frac{x^3}{3} + x + C \]

\[ F(1) = 2 \]

\[ \frac{1}{3} + 1 + C = 2 \]

\[ C = \frac{2}{3} \]

\[ F(x) = \frac{x^3}{3} + x + \frac{2}{3} \]

\[ \int x^2 + 1 \, dx = \frac{4}{3} \]
Chapter 4 Summary

The following list is neither complete nor self-contained. Rather it is meant to trigger your memory and activate your comprehension. If any of these points are not clear to you make sure you review the relevant material before the exam. You want to understand everything that’s indicated here but not everything will be covered by the exam.

• An antiderivative of a function $f$ is any function $F$ such that

$$F' = f.$$  \hspace{1cm} (1)

Throughout these notes we use $F$ to denote an antiderivative of $f$.

• Two antiderivatives of the same function differ by a constant.

• We denote antiderivatives by indefinite integrals:

$$\int f(x)dx = F(x) + C$$  \hspace{1cm} (2)

where $F$ is any particular antiderivative and $C$ is the integration constant. $\int$ is the integration symbol, $x$ is the integration variable, and $f$ is the integrand.

• The integration constant can assume a specific value that is determined by a side condition like $F(0) = 1$. 

We use the **summation symbol**

\[
\sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n = \sum_{j=1}^{n} a_j = \sum_{s=1}^{n} a_s.
\]  

(\(\Sigma\) is the capital Greek letter Sigma, corresponding to S for “sum”). \(i\) (or \(j\) or \(s\)) is the **summation index**. It has no meaning outside the sum, and can be any variable that is not otherwise used in the definition of the sum.

Obvious modifications apply to expressions like

\[
\sum_{k=0}^{n} a_k, \quad \sum_{j=15}^{17} a_j, \quad \text{etc.}
\]

Properties of the summation symbol include

\[
\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i,
\]

\[
\sum_{i=1}^{n} ka_i = k \sum_{i=1}^{n} a_i,
\]

\[
\sum_{i=1}^{n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i.
\]
• Some examples of particular sums include:

\[
\sum_{i=1}^{n} 1 = n
\]

\[
\sum_{i=1}^{n} i = \frac{n(n + 1)}{2}
\]

\[
\sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6}
\]

\[
\sum_{i=1}^{n} i^3 = \left[ \frac{n(n + 1)}{2} \right]^2
\]

• We define **definite integrals** as limits of **Riemann Sums**. A simplified version of that definition is

\[
\int_{a}^{b} f(x) \, dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i) \Delta x
\]

where

\[
\Delta x = \frac{b - a}{n} \quad \text{and} \quad x_i = a + i\Delta x.
\]

• The notation is due to Leibniz. You can think of the integration symbol \( \int \) as a stylish version of the letter \( S \) (for sum). When you take the limit of the Riemann sum then \( \Sigma \) turns into \( \int \), and the quantity \( \Delta x \) turns into \( dx \),
similarly as it did when we defined derivatives.

- One application of this definition is that we can approximate certain objects by sums, recognize these as Riemann sums, take the limit, and so obtain an integral that we can compute.

- In the expression \( \int_a^b f(x)dx \), \( a \) and \( b \) are the lower and upper limits, respectively. As before, \( x \) is the integration variable, and \( f \) is the integrand. The expression \( \int_a^b f(x)dx \) is described in words as the integral of \( f \) with respect to \( x \) from \( a \) to \( b \). Note that the phrase “limits of integration” is unrelated to the technical notion of “limits” that we have discussed in the past, and that we will continue to use in the future.

- We discussed two equivalent versions of the Fundamental Theorem of Calculus.

\[
\int_a^b f(x)dx = F(b) - F(a) \\
\tag{9}
\]

and

\[
\frac{d}{dx} \int_a^x f(t)dt = f(x) \\
\tag{10}
\]
• We also use this notation for definite integrals:

\[
\int_{a}^{b} f(x) \, dx = \left[ F(x) \right]_{a}^{b} = F(b) - F(a). \tag{11}
\]
Properties of the definite integral include

\[
\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx
\]

\[
\int_a^b kf(x)\,dx = k \int_a^b f(x)\,dx
\]

\[
\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx
\]

\[
\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx
\]

\[
\int_{-a}^a f(x)\,dx = 0 \quad \text{if } f \text{ is odd}
\]

\[
\int_{-a}^a f(x)\,dx = 2 \int_0^a f(x)\,dx \quad \text{if } f \text{ is even}
\]

\[
\int_{-r}^r \sqrt{r^2 - x^2}\,dx = \frac{\pi r^2}{2} \quad \text{(area of half a circle)}
\]

\[
\int_0^{2\pi} \sin^2 x\,dx = \int_0^{2\pi} \cos^2 x\,dx = \pi.
\]

(12)

• Usually we compute definite integrals by finding an antiderivative, evaluating it at the limits of integration, and computing the difference of those values, via the fundamental theorem of Calculus. Sometimes, however, we can take shortcuts. For example, the integral of an odd function over an interval centered
Examples

\[ I = \int_0^1 (x^2 + 1) \sqrt{1 - x^2} \, dx \]

\[ \frac{d}{dx} \left( \int_0^x \frac{1}{t^2 + 1} \, dt \right) = \]
at the origin is zero:

\[ \int_{-b}^{b} f(x) \, dx = 0 \quad \text{if } f \text{ is odd} \]

In other cases the integral may be clear from basic geometry. For example

\[ \int_{-r}^{r} \sqrt{r^2 - x^2} \, dx = \frac{\pi r^2}{2} \quad (13) \]

since the graph of the integrand is a semi-circle. Another, somewhat more subtle, example we saw in class was

\[ \int_{a}^{a+2\pi} \sin^2 x \, dx = \int_{a}^{a+2\pi} \cos^2 x \, dx = \pi \quad (14) \]

which follows from the facts that \( \sin \) and \( \cos \) are \( 2\pi \) periodic and just shifts of each other, an integral over one period of a periodic function is independent of the starting point, and the squares of \( \sin \) and \( \cos \) add to 1.

- In general, the limits of integration may be functions of some variable, and we can differentiate with respect to that variable:

\[
\frac{d}{dx} \int_{a(x)}^{b(x)} f(t) \, dt = \frac{d}{dx} \left( F(b(x)) - F(a(x)) \right) \\
= F'(b(x))b'(x) - F'(a(x))a'(x) \\
= f(b(x))b'(x) - f(a(x))a'(x). \quad (15)
\]
• Since differentiation and integration are inverse processes every differentiation rule comes with an integration formula. Just read from right to left, instead of left to right.

• Integrals preserve inequalities: If $f(x) \leq g(x)$ for all $x$ in the interval $[a, b]$ then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$ \hspace{1cm} (16)

• Bounded functions give rise to bounds on integrals: If $m \leq f(x) \leq M$ for all $x$ in the interval $[a, b]$ then

$$m(b - a) \leq \int_a^b f(x)\,dx \leq M(b - a).$$ \hspace{1cm} (17)

• The **Mean Value Theorem for Integrals** says that if $f$ is continuous on $[a, b]$ then there exists a number $c$ in $(a, b)$ such that

$$\int_a^b f(x)\,dx = (b - a)f(c).$$ \hspace{1cm} (18)

• The **average value** of a continuous function $f$ on an interval $[a, b]$ is

$$\text{average} = \frac{\int_a^b f(x)\,dx}{b - a}.$$ \hspace{1cm} (19)
• **Integration by substitution** is the reverse version of the chain rule.

\[
\int f(g(x))g'(x)\,dx = F(g(x)) + C \quad (20)
\]

and

\[
\int_a^b f(g(x))g'(x)\,dx = F(g(b)) - F(g(a)) \quad (21)
\]

• Writing

\[
u = u(x) = g(x) \quad \text{and} \quad du = g'(x)\,dx \quad (22)
\]

we obtain the more common forms

\[
\int f(g(x))g'(x)\,dx = \int f(u)\,du = F(u(x)) + C \quad (23)
\]

and

\[
\int_a^b f(g(x))g'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du = F(u(b)) - F(u(a)). \quad (24)
\]

• Practically speaking this means that when using integration by substitution to compute indefinite integrals, you usually obtain the answer in terms of the original variable, whereas when computing definite integrals you replace both the integrand and the limits of integration, and you don’t return to the original variable.
\[
\frac{1}{2} \int (x+1) \sqrt{1-x^2} \, dx = \frac{1}{2} \int \sqrt{1-x^2} \, dx + \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} \, dx
\]

\[u = 1-x^2\]
\[du = -2x \, dx\]
\[\frac{x^2 + y^2}{y^2} = 1\]
\[y^2 = 1-x^2, \quad y = \sqrt{1-x^2}\]
\[
\int (1-x^2)^{3/2} \, dx = 1-x^2 \]
\[
\frac{d}{dx} (1-x^2)^{3/2} = \frac{3}{2} (1-x^2)^{1/2} \frac{x}{2} \frac{1}{\sqrt{1-x^2}} \frac{x}{2}(1-x^2)^{1/2} \]
\[x = 1\]
\[\int x \sqrt{1-x^2} \, dx = \frac{1}{2} \]

\[u = 1-x^2, \quad u(0) = 1, \quad u(1) = 0\]
\[du = -2x \, dx\]
\[x \, dx = -\frac{1}{2} \, du\]
\[ I = -\frac{1}{2} \int_{1}^{\infty} u^{1/2} \, du = -\frac{1}{2} \cdot \frac{2}{3} \cdot u^{3/2} \Big|_{1}^{\infty} = \frac{1}{3} \cdot 1 = \frac{1}{3} \]

\[ I = \int_{0}^{1} (1 - x^2)^{1/2} \, dx = -\frac{1}{2} \cdot \frac{3}{2} \cdot (1 - x^2)^{3/2} \Big|_{0}^{1} = \frac{1}{3} \]

\[ I = \int_{-1}^{1} (x + 1) \sqrt{1 - x^2} \, dx = \int_{-1}^{1} x \sqrt{1 - x^2} \, dx + \int_{-1}^{1} \sqrt{1 - x^2} \, dx = \frac{\pi}{2} \]

\[ I = \int_{-1}^{1} x \sqrt{1 - x^2} \, dx + \int_{1}^{\infty} \sqrt{1 - x^2} \, dx \]
\[ \lim_{n \to \infty} \sum_{i=1}^{n} \left( x_i^2 + \frac{\pi}{n} \right) \Delta x = \frac{\pi}{4} \sum_{i=1}^{n} x_i^2 + \frac{\pi}{n} \Delta x \]

\[ \Delta x = \frac{\pi}{n} \]

\[ x_i = \pi + i \Delta x \]

\[ i > 0 \implies x_0 = \alpha = \pi + 0 \Delta x = \pi \]

\[ \alpha = x_0 < x_1 < \cdots < x_n = b \]

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