Math 1210-23

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Notes of 4/2-3/24

$$\frac{d}{dx}\int_a^{x^2} f(t)\,dt = f(x^2)\,2x$$

Announcements

- As you know, we will have our exam 3 on

  Friday, **4/5/24**.

- There will be a total of 8 questions on Chapters 3 and 4. see last problem on current hw
- Today and tomorrow we will review Chapters 3 and 4. I'll go over the review, and the rest of the time will be driven by your questions.
- (The review for the final exam will go over several days and will have the same format.)

Most Important Rule

The most important rule in Calculus is not to confuse Differentiation and Integration.

More Rules

- Check all antiderivatives by differentiation.
- Be clear about the variable with respect to which you integrate or differentiate.

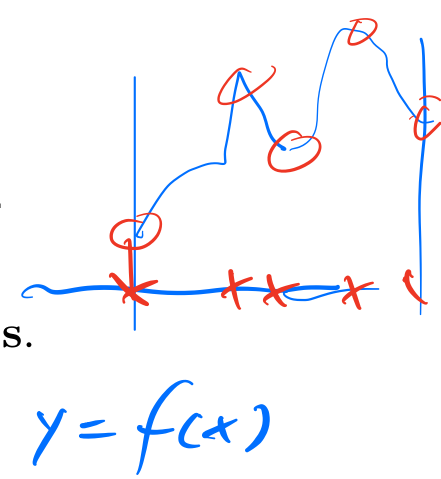
- Go slowly and deliberately.
- Explain what you are doing.

Subject Review

- The following list is neither complete nor self contained. Rather it is meant to trigger your memory and activate your comprehension. If any of these points are not clear to you make sure you review the relevant material before the exam. You want to understand everything that's indicated here but not everything will be covered by the exam.

Chapter 3 Review

- You want to understand what we mean by a **local**, **global**, **absolute**, or **relative**, **minimum**, **maximum**, or **extreme**, **value** of a function.
- The function that you are minimizing or maximizing is sometimes called the **objective function**.
- Extreme values can occur only at **critical points**. These are values of the independent variable. There are three kinds:
  - **endpoints** of intervals,
  - **singular points** where the derivative does not exist,
  - **stationary points** where the derivative is zero.



- Thus a standard procedure for finding extreme values involves identifying all critical points and evaluating the objective function there.
- Of the three kinds, stationary points occur most frequently.
- The function value at a stationary point is
  - a local minimum if the first derivative changes sign from negative to positive,
  - a local minimum if the second derivative is positive,
  - a local maximum if the first derivative changes sign from positive to negative,
  - a local maximum if the second derivative is negative.
- When solving word problems, keep in mind which variables denote constants, which variable denotes the function you want to minimize or maximize, and what is the independent variable. In many word problems, the objective function depends on several variables that are interrelated. In that case you first need to choose one variable as the independent variable and express the others in terms of your chosen independent variable.
- The above criteria are easy to understand and remember if you understand the relationship between derivatives and the shape of a graph.
- A function is
  - **increasing** if the first derivative is posi-

tive,

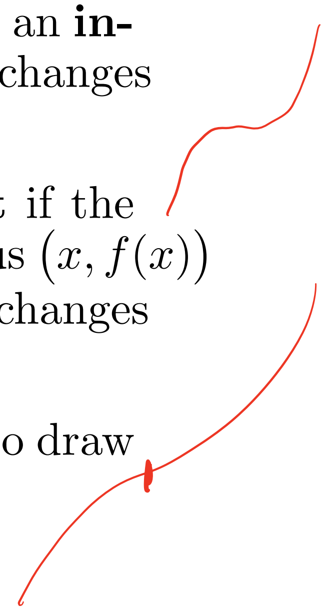
- **decreasing** if the first derivative is negative,
- **concave up** (also sometimes called **convex**) if the second derivative is positive,
- **concave down** (also sometimes called **concave**) if the second derivative is negative.

- The graph of a function has a horizontal tangent at a point where the derivative is zero (naturally).

- A **point of inflection**, or an **inflection point**, is a point on the graph where the graph changes from being concave up to being concave down, or vice versa.

- Thus we have a point of **inflection**, or an **inflection point** if the second derivative changes sign.

$y = x^3$



- An inflection point is a **saddle point** if the tangent at that point is horizontal. Thus $(x, f(x))$ is a saddle point if $f'(x) = 0$ and $f''(x)$ changes sign.

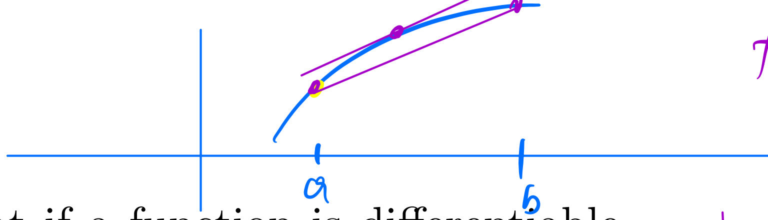
- You want to be able to use these facts to draw the graph of a function.

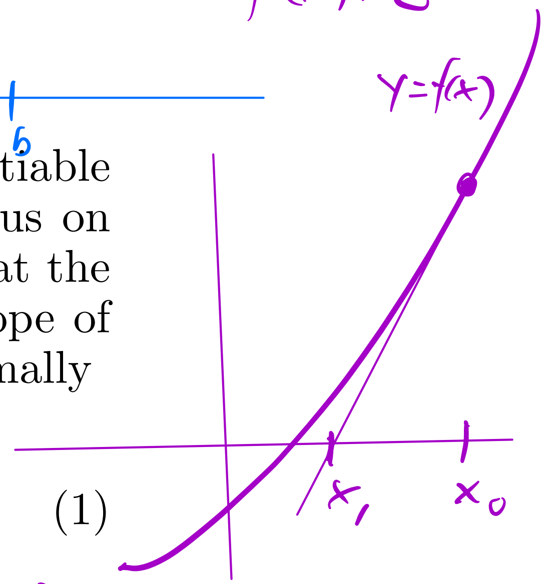
- Use **redundancy** to check your work.

- You also want to be able to check that these facts are consistent with the graph of a specific function.

- The **mean value theorem** (MVT) (for deriva-

$f(x)=0$





tives) says that if a function is differentiable in an open interval (a,b) and continuous on $[a,b]$ there is a point c in (a,b) such that the slope of the tangent at c equals the slope of the secant over the entire interval. Formally

$$f'(c) = \frac{f(b)-f(a)}{b-a}. \tag{1}$$

$f(b) - f(a) = f'(c)(b-a)$

- To solve $f(x) = 0$ we can use **Newton's Method**:

$$x_0 \quad \text{given}, \qquad x_{k+1} = g(x_k) \quad \text{where} \quad g(x) = x - \frac{f(x)}{f'(x)}.$$

This is a special case of a **fixed point iteration**.

$f(x) = 0 \iff x = g(x)$

Chapter 4 Summary

- An **antiderivative** of a function f is any function F such that

$$\boxed{F' = f.} \tag{2}$$

$f(x) = \cos x$
$F(x) = \sin x + C$
$(\sin x) + \pi$

Throughout these notes we use F to denote an antiderivative of f.

- Two antiderivatives of the same function differ by a constant.

- We denote antiderivatives by **indefinite integrals:**

$$\int f(x)\mathrm{d}x = F(x) + C \tag{3}$$

where F is any particular antiderivative and C is the **integration constant.** $\int$ is the **integration symbol**, x is the **integration variable**, and f is the **integrand**.

- The integration constant can assume a specific value that is determined by a side condition like $F(0) = 1$.

- We use the **summation symbol**

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n = \sum_{j=1}^{n} a_j = \sum_{s=1}^{n} a_s. \tag{4}$$

(Σ is the capital Greek letter Sigma, corresponding to S for "sum"). i (or j or s) is the **summation index**. It has no meaning outside the sum, and can be any variable that is not otherwise used in the definition of the sum.

- Obvious modifications apply to expressions like

$$\sum_{k=0}^{n} a_k, \quad \sum_{j=15}^{17} a_j, \quad \text{etc.} \tag{5}$$

- Properties of the summation symbol include

$$\begin{aligned}
\sum_{i=1}^{n}(a_i+b_i) &= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i,\\
\sum_{i=1}^{n} k a_i &= k\sum_{i=1}^{n} a_i,\\
\sum_{i=1}^{n} a_i &= \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i.
\end{aligned} \tag{6}$$

- Some examples of particular sums include:

$$\begin{aligned}
\sum_{i=1}^{n} 1 &= n\\
\sum_{i=1}^{n} i &= \frac{n(n+1)}{2}\\
\sum_{i=1}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6}\\
\sum_{i=1}^{n} i^3 &= \left[\frac{n(n+1)}{2}\right]^2
\end{aligned} \tag{7}$$

- We define **definite integrals** as limits of **Riemann Sums**. A simplified version of that definition is

$$\int_a^b f(x)\mathrm{d}x = \lim_{\Delta x \longrightarrow 0} \sum_{i=1}^{n} f(x_i)\Delta x \tag{8}$$

where

$$\Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i\Delta x. \tag{9}$$

- The notation is due to Leibniz. You can think of the integration symbol $\int$ as a stylish version of the letter S (for sum). When you take the limit of the Riemann sum then Σ turns into $\int$, and the quantity Δx turns into $\mathrm{d}x$, similarly as it did when we defined derivatives.

- One application of this definition is that we can approximate certain objects by sums, recognize these as Riemann sums, take the limit, and so obtain an integral that we can compute.

- In the expression $\displaystyle\int_a^b f(x)\mathrm{d}x$, a and b are the **lower** and **upper** limits, respectively. As before, x is the **integration variable**, and f is the **integrand.** The expression $\int_a^b f(x)\mathrm{d}x$ is described in words as *the integral of f with respect to x from a to b.* Note that the phrase "limits of integration" is unrelated to the technical notion of "limits" that we have discussed in the past, and that we will continue to use in the future.

- We discussed two equivalent versions of the **Fundamental Theorem of Calculus.**

$$\boxed{\int_a^b f(x)\mathrm{d}x = F(b) - F(a)} \qquad (10)$$

and

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\mathrm{d}t = f(x)}. \qquad (11)$$

- We also use this notation for definite integrals:

$$\int_a^b f(x)\mathrm{d}x = \Big[F(x)\Big]_a^b = F(x)\Big|_a^b. \qquad (12)$$

- **Properties** of the definite integral include

$$
\begin{aligned}
&\int_a^b f(x)+g(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}x \\
&\int_a^b kf(x)\mathrm{d}x = k\int_a^b f(x)\mathrm{d}x \\
&\int_a^b f(x)\mathrm{d}x = -\int_b^a f(x)\mathrm{d}x \\
&\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x \\
&\int_{-a}^a f(x)\mathrm{d}x = 0 \qquad \text{if } f \text{ is odd} \\
&\int_{-a}^a f(x)\mathrm{d}x = 2\int_0^a f(x)\mathrm{d}x \qquad \text{if } f \text{ is even} \\
&\int_{-r}^r \sqrt{r^2-x^2}\mathrm{d}x = \frac{\pi r^2}{2} \qquad \text{(area of half a circle)} \\
&\int_0^{2\pi} \sin^2 x\mathrm{d}x = \int_0^{2\pi}\cos^2 x\mathrm{d}x = \pi.
\end{aligned}
\tag{13}
$$

- Usually we compute definite integrals by finding an antiderivative, evaluating it at the limits of integration, and computing the difference of those values, via the fundamental theorem of Calculus. Sometimes, however, we can take shortcuts. For example, the integral of an odd function over an interval centered at the origin is zero:

$$\int_{-b}^{b} f(x)\mathrm{d}x = 0 \qquad \text{if } f \text{ is odd}$$

In other cases the integral may be clear from basic geometry. For example

$$\int_{-r}^{r} \sqrt{r^2 - x^2}\mathrm{d}x = \frac{\pi r^2}{2} \tag{14}$$

since the graph of the integrand is a semi circle. Another, somewhat more subtle, example we saw in class was

$$\int_{a}^{a+2\pi} \sin^2 x \mathrm{d}x = \int_{a}^{a+2\pi} \cos^2 x \mathrm{d}x = \pi \tag{15}$$

which follows from the facts that sin and cos are 2π periodic and just shifts of each other, an integral over one period of a periodic function is independent of the starting point, and the squares of sin and cos add to 1.

- In general, the limits of integration may be functions of some variable, and we can differentiate with respect to that variable:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} \int_{a(x)}^{b(x)} f(t)\mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}x} \Big(F\big(b(x)\big) - F\big(a(x)\big) \Big) \\ &= F'\big(b(x)\big)b'(x) - F'\big(a(x)\big)a'(x) \\ &= f\big(b(x)\big)b'(x) - f\big(a(x)\big)a'(x). \end{aligned} \tag{16}$$

- Since differentiation and integration are inverse processes every differentiation rule comes with an integration formula. Just read from right to left, instead of left to right.

- integrals preserve inequalities: If $f(x) \le g(x)$ for all x in the interval $[a,b]$ then

$$\int_a^b f(x)\mathrm{d}x \le \int_a^b g(x)\mathrm{d}x. \tag{17}$$

- Bounded functions give rise to bounds on integrals: If $m \le f(x) \le M$ for all x in the interval $[a,b]$ then

$$m(b-a) \le \int_a^b f(x)\mathrm{d}x \le M(b-a). \tag{18}$$

- The **Mean Value Theorem for Integrals** says that if f is continuous on $[a,b]$ then there exists a number c in (a,b) such that

$$\int_a^b f(x)\mathrm{d}x = (b-a)f(c). \tag{19}$$

- The **average value** of a continuous function f on an interval $[a,b]$ is

$$\text{average} = \frac{\int_a^b f(x)\mathrm{d}x}{b-a}. \tag{20}$$

- **Integration by substitution** is the reverse version of the chain rule.

$$\int f\big(g(x)\big)g'(x)\mathrm{d}x = F\big(g(x)\big) + C \tag{21}$$

and

$$\int_a^b f\big(g(x)\big)g'(x)\mathrm{d}x = F\big(g(b)\big) - F\big(g(a)\big) \tag{22}$$

- Writing

  $$u = u(x) = g(x) \quad \text{and} \quad \mathrm{d}u = g'(x)\mathrm{d}x \tag{23}$$

  we obtain the more common forms

  $$\int f\big(g(x)\big)g'(x)\mathrm{d}x = \int f(u)\mathrm{d}u = F(u(x)) + C \tag{24}$$

  and

  $$\int_a^b f\big(g(x)\big)g'(x)\mathrm{d}x = \int_{u(a)}^{u(b)} f(u)\mathrm{d}u = F\big(u(b)\big) - F\big(u(a)\big). \tag{25}$$

- Practically speaking this means that when using integration by substitution to compute indefinite integrals, you usually obtain the answer in terms of the original variable, whereas when computing definite integrals you replace both the integrand and the limits of integration, and you don't return to the original variable.