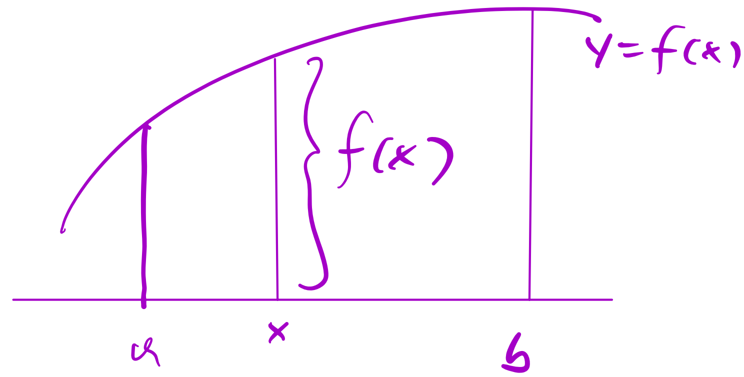
Math 1210-23

Notes of 4/1/24



5.2-5.3 Computation of Volumes

- It will take more than one meeting to cover this set of notes ...
- Sections 5.2-5.3 covers the computation of volumes.
- The basic idea is to integrate the area of a cross-section, in a direction perpendicular to the cross section.
- The textbook stresses another way of thinking about this:

Slice, Approximate, Integrate

- In general it's a good idea first to try a new concept in a familiar context.
- We know that the volume of a sphere of radius r is
$$V = \frac{4\pi r^3}{3}.$$
- How do we actually know this?

- Here is an idea:

1. Slice the sphere into n circular slices (slabs) of equal thickness.
2. Approximate the volume of each slice by thinking of it as a cylinder.
3. Add the volumes of those slices.
4. Take the limit as the number of slices goes to infinity and their thickness goes to zero.
5. Recognize this as the limit of a Riemann Sum, i.e., an integral.
6. Evaluate the integral.

- Here we go …

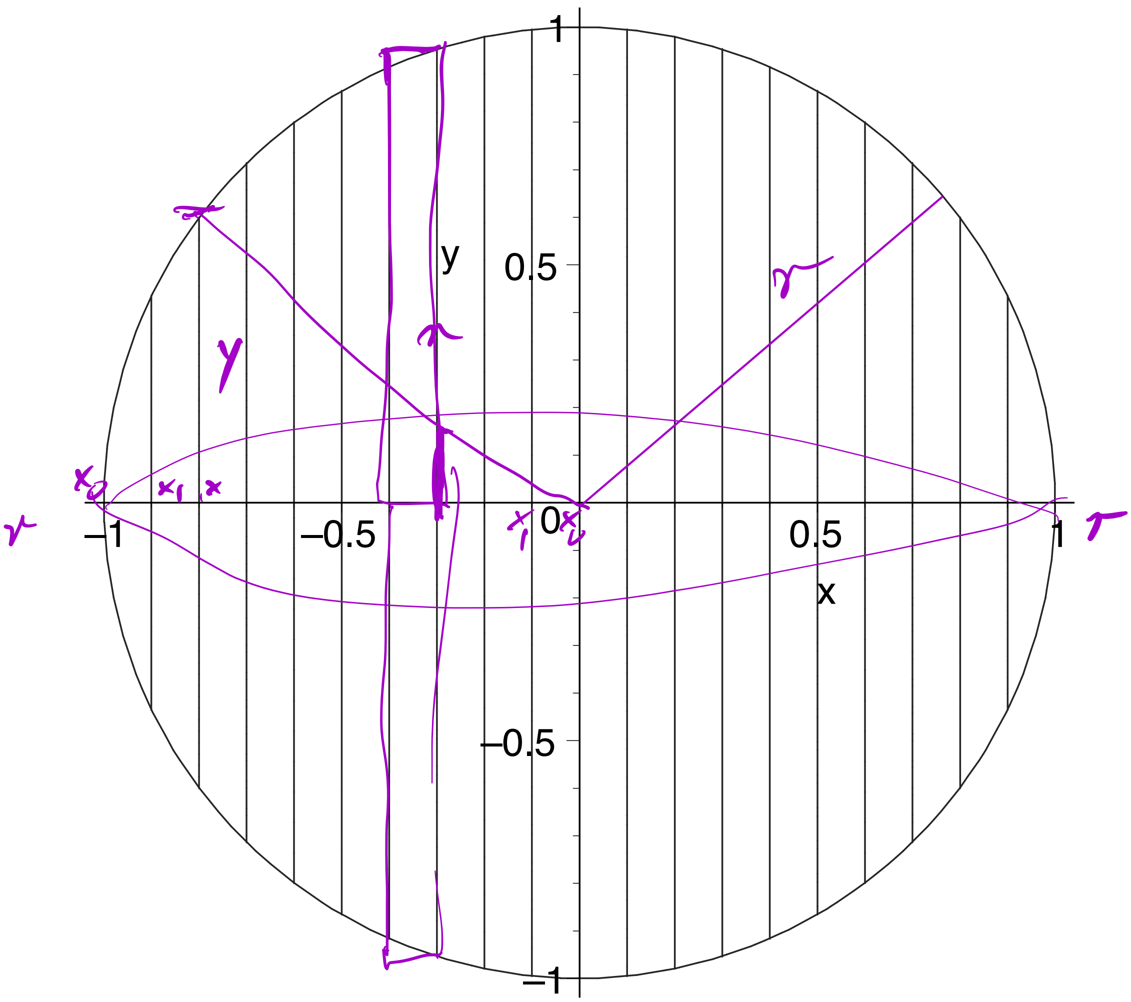


Figure 1. Slicing the Sphere.

- We get

$$\Delta x = \frac{2r}{n} \quad \text{and} \quad x_i = -r + i\Delta x$$

- The radius of the circular slice from x_{i-1} to x_i is approximately

$$r_i = \sqrt{r^2 - x_i^2}$$

and its volume is approximately

$$V_i = \pi r_i^2 \Delta x = \pi(r^2 - x_i^2)\Delta x$$

- The total volume of the sphere, therefore, is

$$\begin{aligned} V &= \lim_{n \longrightarrow \infty} \sum_{i=1}^{n} V_i \\ &= \lim_{n \longrightarrow \infty} \sum_{i=1}^{n} \pi(r^2 - x_i^2)\Delta x \\ &= \int_{-r}^{r} \pi(r^2 - x^2)\mathrm{d}x \\ &= \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r} \\ &= \pi \left[r^3 - \frac{r^3}{3} - \left(-r^3 + \frac{r^3}{3} \right) \right] = \pi\left(2r^3 - \frac{2}{3}r^3\right) \\ &= \frac{4\pi r^3}{3} = \frac{4}{3}\pi r^3 \end{aligned}$$

- Note that we integrate the area of the cross section!

- The textbook calls this approach the **Method of Disks**.

- Here is an alternative approach, called the **Method of Shells**

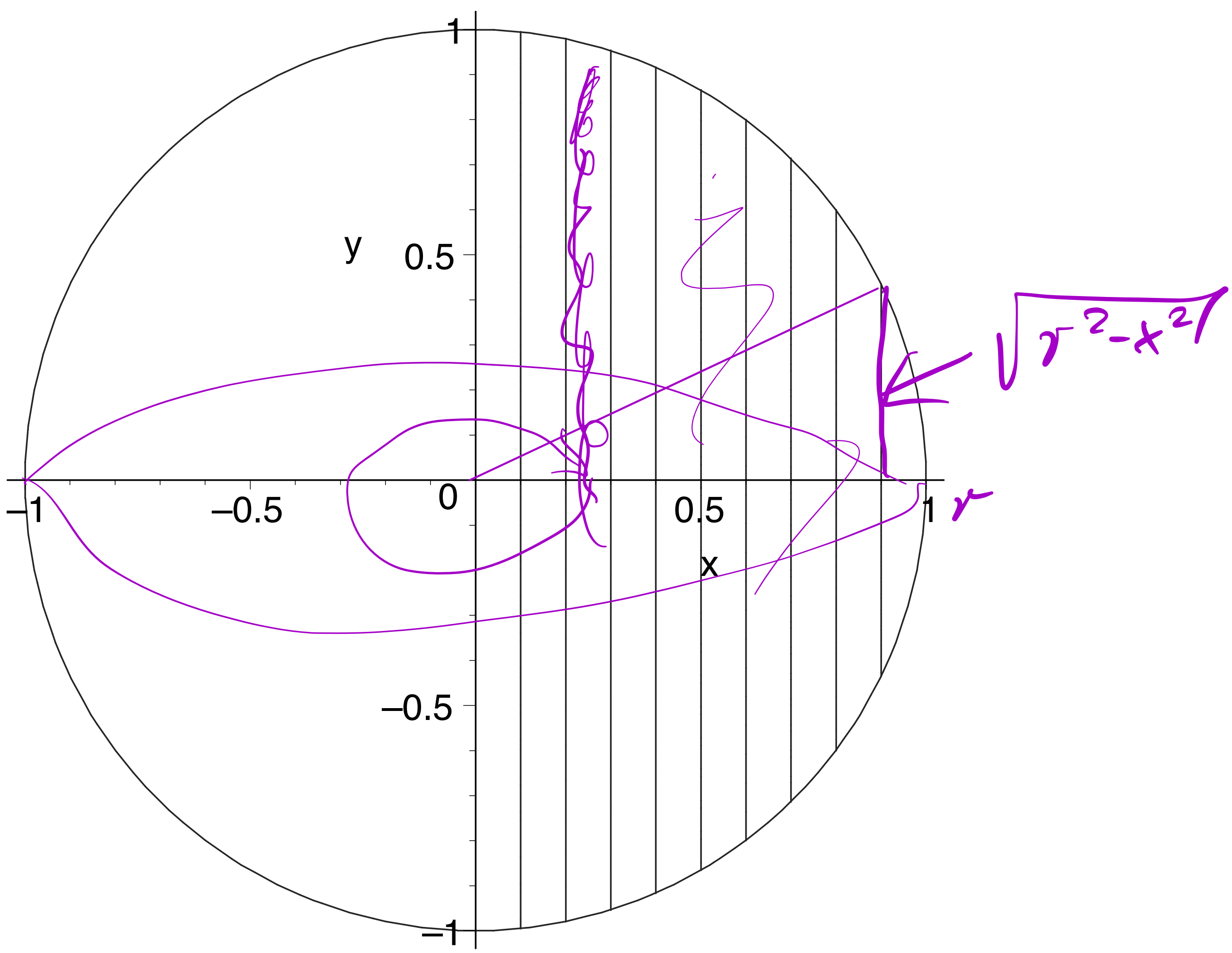


Figure 2. Slicing the Sphere, Again.

- Think of the sphere as having been obtained by rotating a circle around the y-axis. Slice the right side of this circle as before. Then rotating each of the vertical regions in Figure 2 forms a circular shell (a cylindrical wall). We want to add up the volumes of these shells and take the limit, as before.

- Again, we need some notation. Let

$$\Delta x = \frac{r}{n} \quad \text{and} \quad x_i = i\Delta x.$$

  Then the radius of each of these shells is x_i, its circumference is $2\pi x_i$, its height is $h_i = 2\sqrt{r^2 - x_i^2}$ and its thickness is Δx.

- Its volume V_i equals approximately height times circumference times thickness, i.e.,

$$V_i = 4\pi x_i \sqrt{r^2 - x_i^2}\Delta x.$$

- Summing and taking the limit as before gives:

$$\begin{aligned} V &= \lim_{n \longrightarrow \infty} \sum_{i=1}^{n} 4\pi x_i \sqrt{r^2 - x_i^2}\Delta x \\ &= 4\pi \int_0^r x\sqrt{r^2 - x^2}\, x \mathrm{d}x \\ &= 4\pi \int_0^r x(r^2 - x^2)^{1/2} \mathrm{d}x \\ &= 4\pi \left[-\frac{2}{3} \times \frac{1}{2} (r^2 - x^2)^{\frac{3}{2}} \right]_0^r \\ &= \frac{4\pi r^3}{3} \end{aligned}$$

as before.

- Again, we integrate the area of the cross section.

Volume of a Paraboloid

- Example 1, page 283. Compute the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}$, the x-axis, and the line $x = 4$ around the x-axis.

$$V_{cyl} = \pi r^2 4 = \pi 2^2 \cdot 4 = 16\pi$$

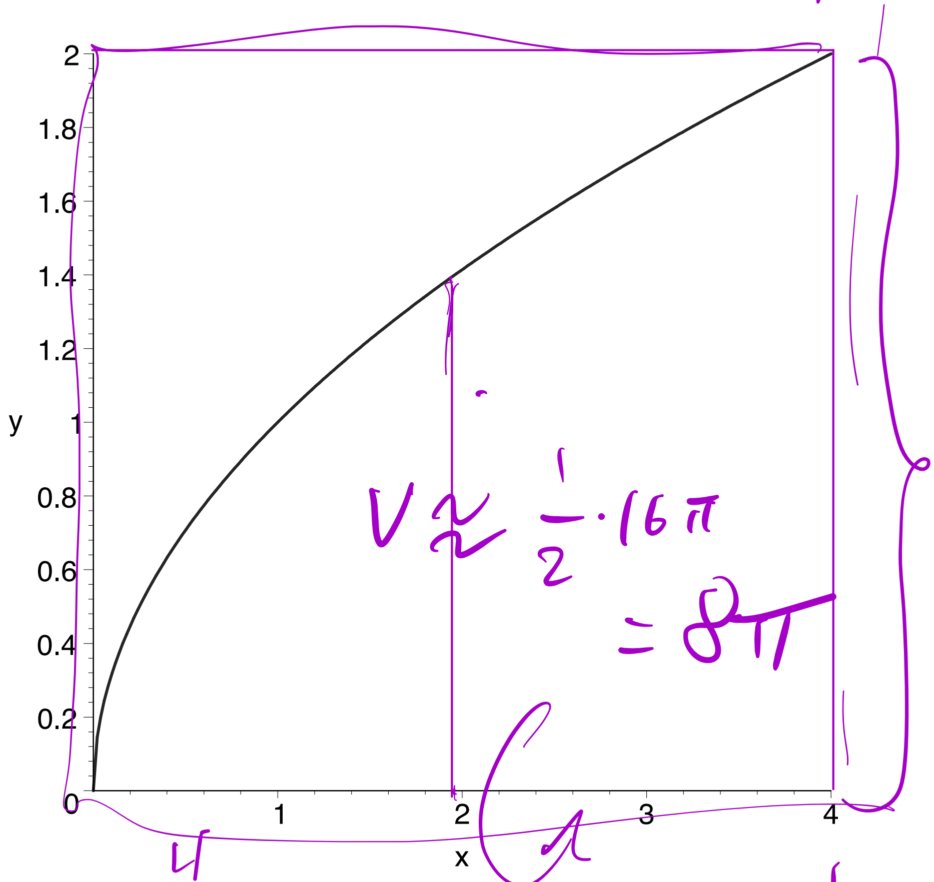


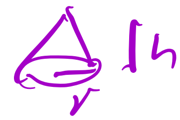
Figure 3. A Paraboloid.

$$V = \int_0^4 \pi y^2 \, dx = \pi \int_0^4 x \, dx = \pi \left[\frac{x^2}{2} \right]_0^4$$

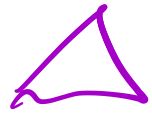
$$y = \sqrt{x} \qquad y^2 = x$$

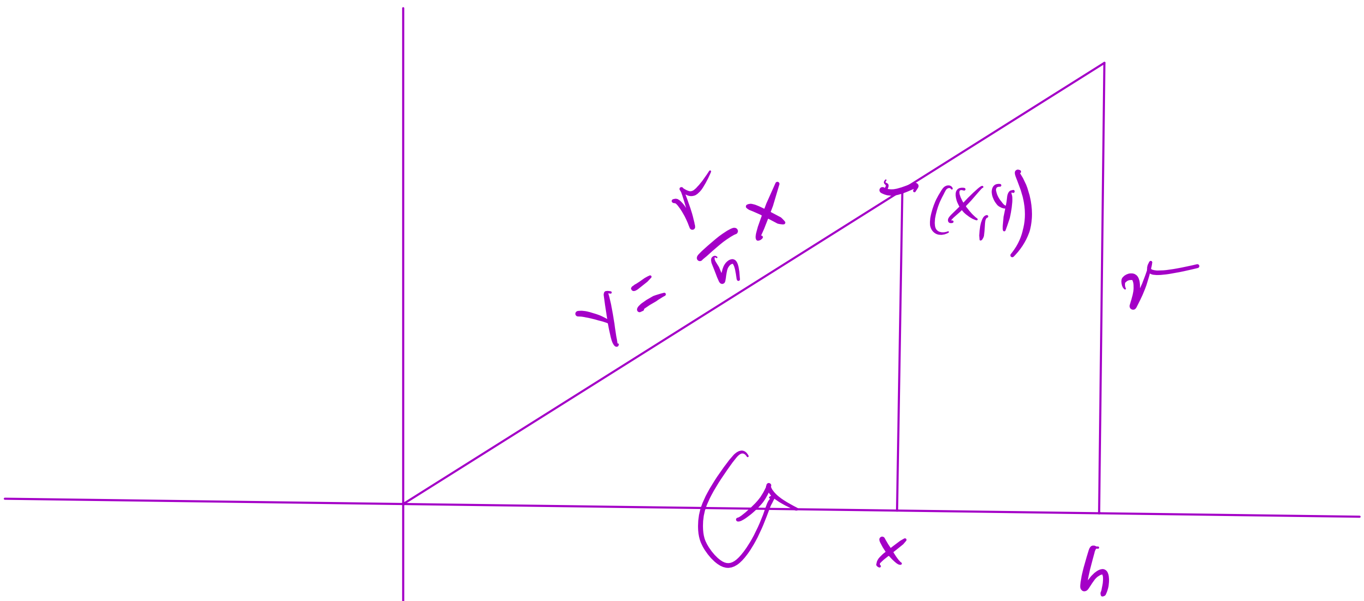
$$= \pi (8 - 0) = 8\pi$$

Volume of a Cone



$$V_{cone} = \frac{\pi r^2 h}{3}$$





$$V_{cone} = \int_0^h \pi y^2 \, dx$$

$$= \int_0^h \pi \frac{r^2}{h^2} x^2 \, dx$$

$$= \frac{\pi r^2}{h^2} \int_0^h x^2 \, dx = \frac{\pi r^2}{h^2} \left[\frac{x^3}{3} \right]_0^h$$

$$= \frac{\pi r^2}{h^2} \left(\frac{h^3}{3} - 0 \right) = \frac{\pi r^2 h}{3}$$

- Example 6, page 286. The base of a solid is the region between one arc of $y = \sin x$ and the cross section perpendicular to the x axis and parallel to the y-axis is an equilateral triangle. Find the volume of that solid.
- See Figure 14 on page 286 of the textbook.

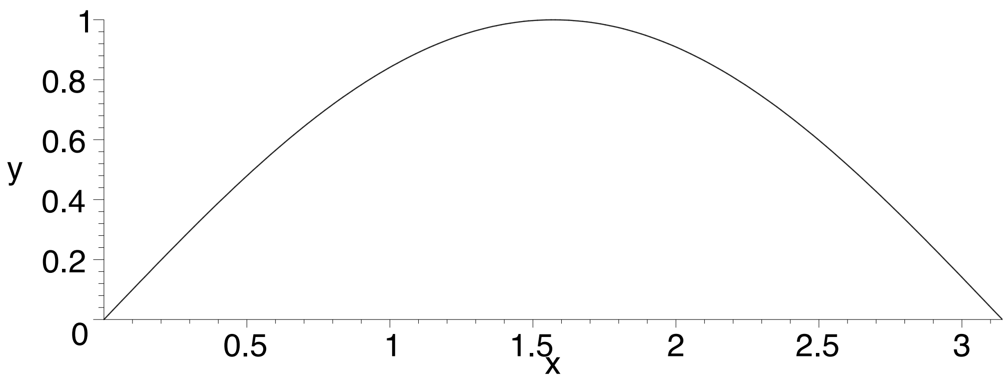


Figure 4. Base of Solid.

- Tour de Force, Example 4, page 291. "Putting it All Together".

- Let R be the region bounded by the y axis, the curve $y = 3 + 2x - x^2$, and the x-axis. Compute the volume of the solid obtained by rotating this region around

a) the x-axis

b) The y-axis

c) the line $y = -1$

d) the line $x = 4$.

- We will skip the actual computation of the integrals, and just set up the integrals.

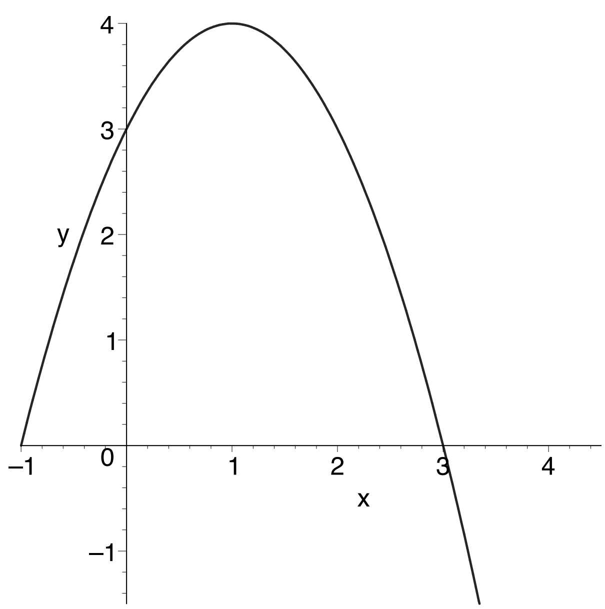


Figure 5. That Region.

- Note that in all 4 cases we will be integrating in x running from 0 to 3.

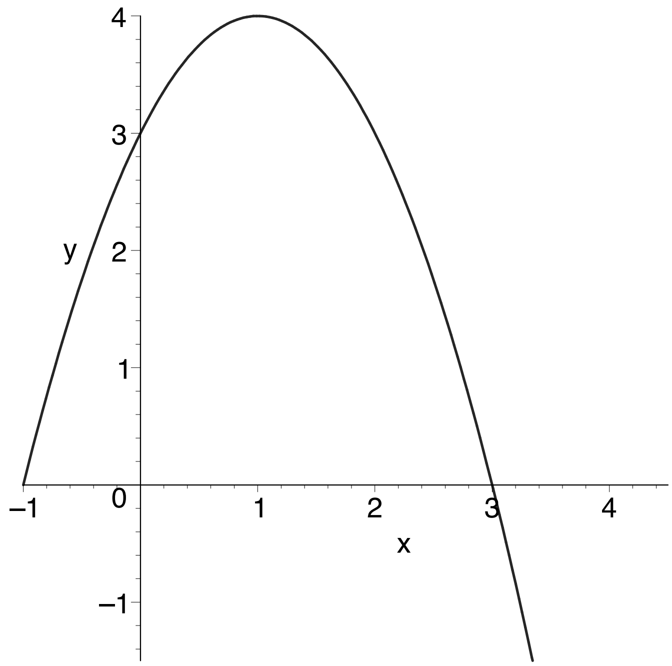


Figure 6. Rotating around the x-axis.

$$\begin{aligned} V &= \int_0^3 \pi y^2 \mathrm{d}x \\ &= \int_0^3 \pi \left(3 + 2x - x^2\right)^2 \mathrm{d}x \\ &= \frac{153\pi}{5} \end{aligned}$$

- This is an example of the method of disks.

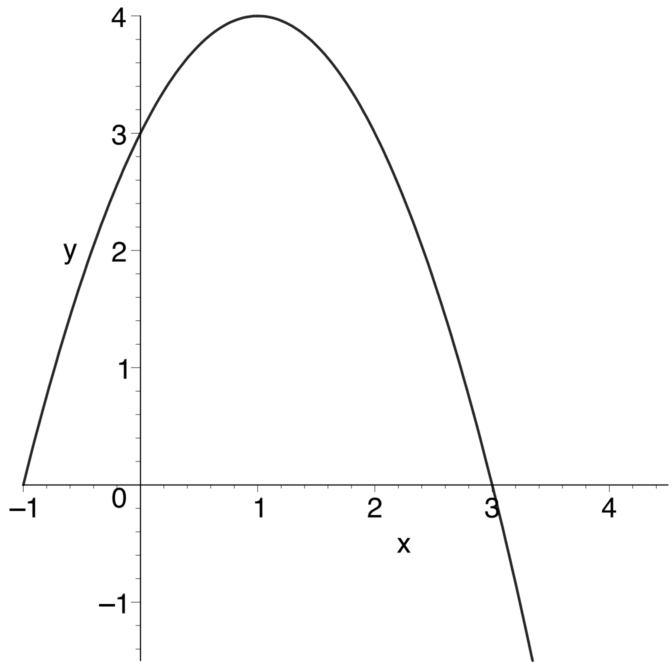


Figure 7. Rotating around the y-axis.

$$
\begin{aligned}
V &= \int_0^3 y \times 2\pi x \mathrm{d}x \\
&= \int_0^3 2\pi x(3 + 2x - x^2)\mathrm{d}x \\
&= \frac{45\pi}{2}
\end{aligned}
$$

- This is an example of the method of shells.

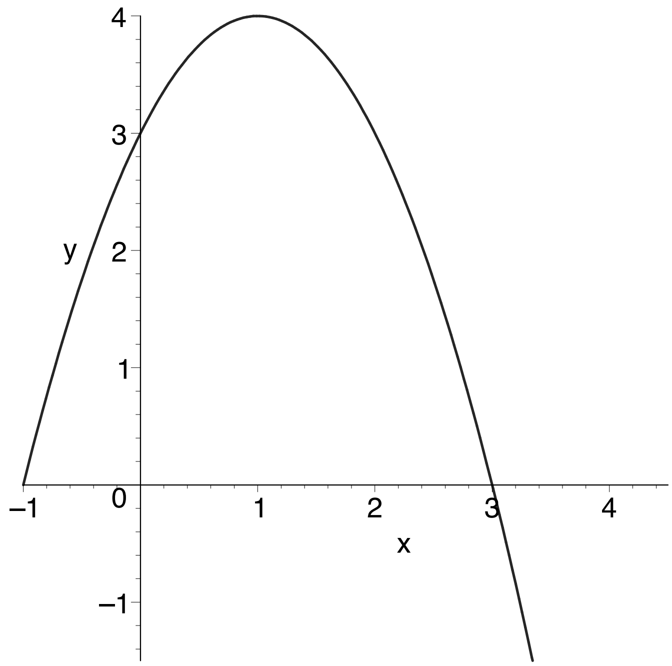


Figure 8. Rotating around the line $y = -1$.

$$V = \int_0^3 \pi\Big((y+1)^2 - 1^2\Big)\mathrm{d}x$$
$$= \pi \int_0^3 (4+2x-x^2)^2 - 1\mathrm{d}x$$
$$= \frac{243\pi}{5}$$

- The textbook calls this the method of washers.

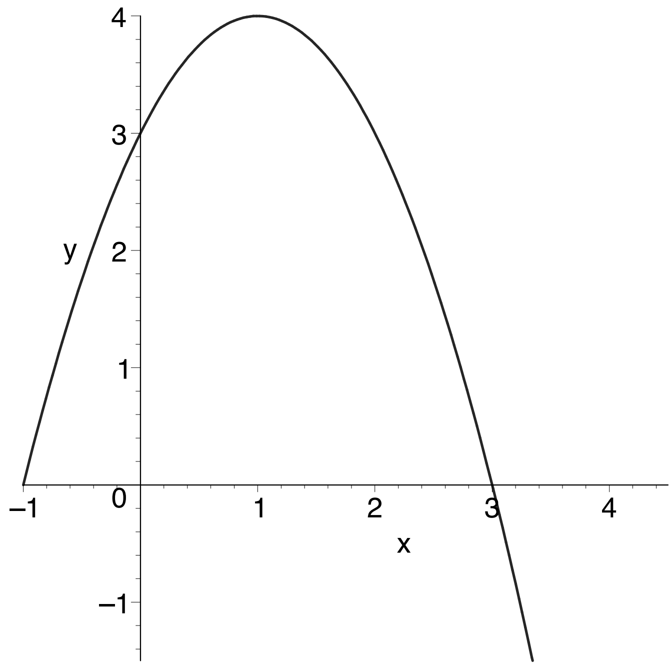


Figure 9. Rotating around the line $x = 4$.

$$V = \int_0^3 2\pi(4-x)y\mathrm{d}x$$
$$= \int_0^3 2\pi(4-x)(3+2x-x^2)\mathrm{d}x$$
$$= \frac{99\pi}{2}$$

- This is another example of the method of shells.