4.3-4 More on the FToC

- Recall the two (easily seen to be equivalent) versions of the Fundamental Theorem of Calculus:

\[
\frac{d}{dx} \int_a^x f(t)\,dt = f(x) \quad \text{and} \quad \int_a^b f(x)\,dx = F(b) - F(a)
\]

where, throughout today, and most of the time in general, \( F \) is an antiderivative of \( f \):

\[
F' = f
\]

- We have the following notations:

\[
\int_a^b f(x)\,dx = F(b) - F(a) = [F(x)]_a^b = [F(x)]_{x=a}^{x=b} = F(x)|_a^b = F(x)|_{x=a}^{x=b}
\]
Some Warmup Examples

\[ \int_0^1 x^p \, dx = \frac{x^{p+1}}{p+1} \bigg|_0^1 = \frac{1}{p+1} - 0 = \frac{1}{p+1} \]

\[ \frac{d}{ds} \int_a^s \sin t^2 \, dt = \sin s^2 = \frac{d}{ds} \left( F(s) - F(a) \right) \]
\[ F'(s) = \sin t^2 \]

\[ \frac{d}{ds} \int_a^s \sin t^2 \, dt = \frac{d}{ds} \left( F(s) - F(a) \right) = \sin (s^2) \cos s^2 \]

\[ \int_0^2 x \, dt = xt \bigg|_0^2 = 2x - 0 \times = 2x \]

\[ \int c \, dx = c \times \]

\[ \frac{d}{dx} \int_0^T t \, dt = 0 \]
More General Differentiation

\[
\frac{d}{dx} \int_{a(x)}^{b(x)} f(t) \, dt = \frac{d}{dx} \left( F(b(x)) - F(a(x)) \right)
\]

\[
= F'(b(x)) b'(x) - F'(a(x)) a'(x)
\]

\[
= f(b(x)) b'(x) - f(a(x)) a'(x)
\]

• Example

\[
\frac{d}{dx} \int_{\sin x}^{\cos x} \sin t \, dt = -\left( \sin(\cos x) \right) \sin x - \left( \sin(\sin x) \right) \cos x
\]
• We started by thinking of definite integrals as areas. That interpretation requires the integrand to be non-negative.

• \( \int_{0}^{1} x \, dx = \frac{x^2}{2} \bigg|_{0}^{1} = \frac{1}{2} \)

• But consider instead:

• \( \int_{0}^{1} -x \, dx = -\frac{x^2}{2} \bigg|_{0}^{1} = -\frac{1}{2} - 0 = -\frac{1}{2} \)

• We get the negative of the area. Regions underneath the \( x \)-axis have a “negative area”.

• \( \int_{0}^{2\pi} \sin x \, dx = -\cos x \bigg|_{0}^{2\pi} = -\cos 2\pi + \cos 0 = 0 \)

• The areas above and below the \( x \) axis cancel.
• Note that we don’t have to interpret the integral as area. For example, if \( f(t) \) gives the velocity at time \( t \) then the integral is distance. This is because for constant velocity distance equals time times velocity.
Switching Limits of Integration

\[ \int_1^2 x^2 + x \, dx = \frac{x^3}{3} + \frac{x^2}{2} \bigg|_1^2 = \frac{8}{3} + 2 - \left(\frac{1}{3} + \frac{1}{2}\right) \]
\[ = \frac{7}{3} - \frac{3}{2} = \frac{5}{6} \]

\[ \int_2^1 x^2 + x \, dx = \frac{x^3}{3} + \frac{x^2}{2} \bigg|_2^1 = \left(\frac{1}{3} + \frac{1}{2}\right) - \left(\frac{8}{3} + 2\right) = -\frac{5}{6} \]

• in general

\[ \int_a^b f(x) \, dx = F(b) - F(a) \]
\[ = -(F(a) - F(b)) \]
\[ = -\int_b^a f(x) \, dx \]

\[ \int_a^b f(x) \, dx = -\int_b^a f(x) \, dx \]
• We discussed

\[ \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \]

• This makes geometric sense if

\[ a < c < b. \tag{1} \]

• It also follows from the FToC:

\[ \int_a^b f(x)\,dx = F(b) - F(a) = F(c) - F(\alpha) + F(6) - F(c) \]

\[ = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \]

• However, we don’t need the assumption (1).

• For example:

\[ \frac{1}{3} = \frac{x^3}{3} \bigg|_0^1 = \int_0^1 x^2\,dx = \int_0^2 x^2\,dx + \int_2^1 x^2\,dx. \]

\[ = \frac{x^3}{3} \bigg|_0^2 + \frac{x^3}{3} \bigg|_2^1 = \frac{8}{3} - 0 + \frac{1}{3} - \frac{8}{3} = \frac{1}{3} \]

• Geometric interpretation:
Integrals May Not Exist

Example

\[ \int_0^1 \frac{1}{x^2} \, dx \]

does not exist, even though we can evaluate the antiderivative at the limits of integration and take the difference.

\[ \begin{array}{c|c|c|c|c|c|c|c} 
 & -1 & -0.8 & -0.6 & -0.4 & -0.2 & 0 & 0.2 \ \hline 
 y & 10 & 8 & 6 & 4 & 2 & 0 & 2 \\
 \end{array} \]

\[ \begin{array}{c|c|c|c|c|c|c} 
 & -1 & -0.8 & -0.6 & -0.4 & -0.2 & 0 & 0.2 \ \hline 
 x & -1 & -0.8 & -0.6 & -0.4 & -0.2 & 0 & 0.2 \\
 \end{array} \]

**Figure 1.** Graph of \( y = 1/x^2 \).

- However, all continuous functions are integrable.
- Many discontinuous functions are too, but we leave this to another day.
Comparison Property

• See page 235, textbook.

• Which is larger, \( \int_0^{10} x^2 \, dx \), or \( \int_0^{10} x^2 + 1 \, dx \)?

• In general: Suppose \( f(x) \leq g(x) \) for all \( x \) in \([a, b]\) (where \( a < b \)). Then

\[
\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx.
\]

(See Theorem B, page 235).

• This follows, for example, from the Riemann Sum definition of the integral.

• Consequence: Suppose \( m \leq f(x) \leq M \) for all \( x \) in \([a, b]\). Then

\[
\int_a^b m \, dx \leq \int_a^b f(x) \, dx \leq \int_a^b M \, dx,
\]

i.e.,

\[
m(b - a) \leq \int_a^b f(x) \, dx \leq M(b - a).
\]

(See Theorem C, page 236).
Linearity

- Major Property (See Theorem D, page 236).

\[ \int_a^b k f(x) \, dx = k \int_a^b f(x) \, dx \]

and

\[ \int_a^b f(x) + g(x) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx. \]
Integration by Substitution

- Inverse Process of the Chain Rule
- \( \int x \sin x^2 \, dx = \)

\[
\int f(g(x)) g'(x) \, dx = f(g(x)) + C
\]

\[ u = x^2 \]
\[ \frac{du}{dx} = 2x \]
\[ du = 2x \, dx \]
\[ x \, dx = \frac{1}{2} \, du \]

\[
\int x \sin x^2 \, dx = \frac{1}{2} \int \sin u \, du = -\frac{1}{2} \cos u
\]
\[ = -\frac{1}{2} \cos x^2 \]

\[
\frac{d}{dx} \left( -\frac{1}{2} \cos x^2 \right) = -\frac{1}{2} \left( -\sin x^2 \right) 2x = x \sin x^2\]
• Example 11, page 247

• \[ \int_0^{\pi/4} \sin^3 2x \cos 2x \, dx = \int \]

\[
\begin{align*}
& u = \sin 2x \\
& du = 2 \cos 2x \, dx \\
& \cos 2x \, dx = \frac{1}{2} \, du
\end{align*}
\]

\[ \int \sin^3 2x \cos 2x \, dx = \frac{1}{2} \int u^3 \, du \]

\[ = \frac{1}{2} \frac{u^4}{4} + C \]

\[ = \frac{1}{2} \frac{\sin^4 2x}{4} \]

\[ I = \frac{1}{2} \frac{\sin^4 2x}{4} \left[ \frac{\pi}{4} \right] = \frac{1}{8} \]

\[ u = 1 = \sin \frac{\pi}{2} \]

\[ \int_0^{\pi/4} \sin^3 2x \cos 2x \, dx \]

\[ x = 0 \]

\[ = \frac{1}{8} \]
• Example 12, page 248

• \[ \int_0^1 \frac{x+1}{(x^2+2x+6)^2} \, dx = \]