Math 1210-23

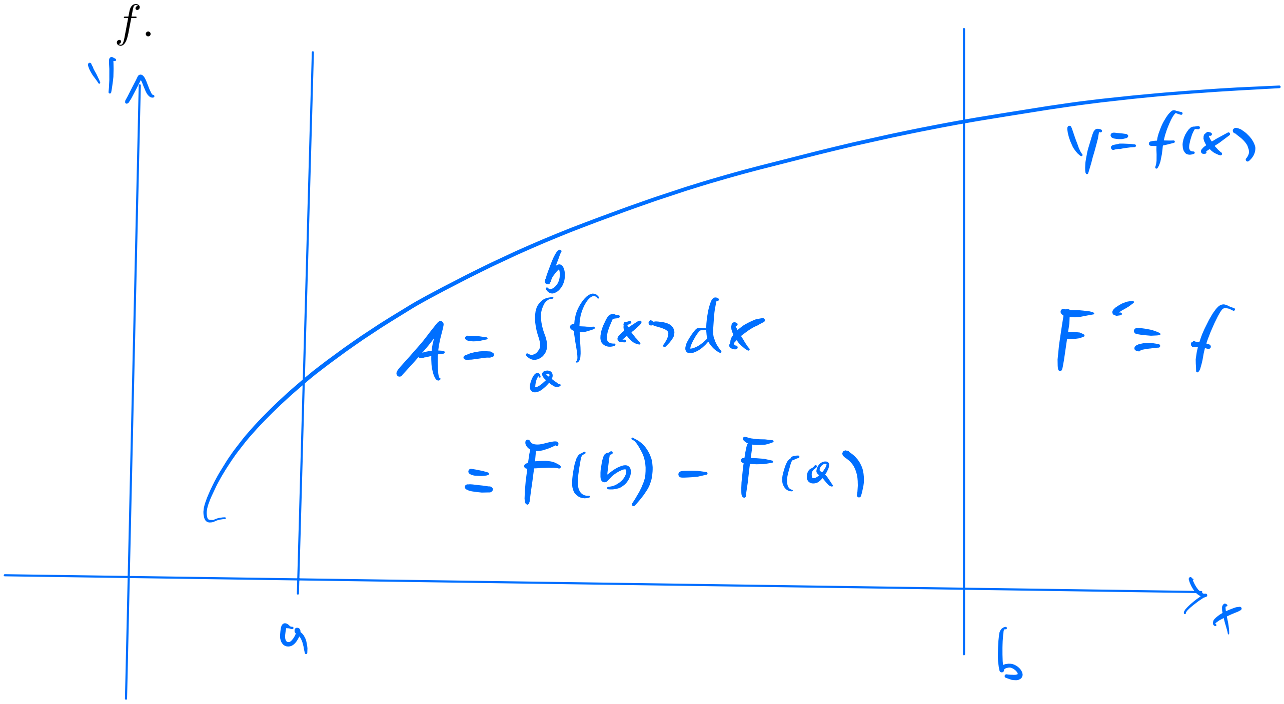
Notes of 3/29/24

5.1 The Area of a Plane Region

- We saw that if $f(x) \geq 0$ for all x in $[a, b]$ then

$$A = \int_a^b f(x) \mathrm{d}x$$

  is the area of the region enclosed by the x-axis, the lines $x = a$ and $x = b$, and the graph of f.

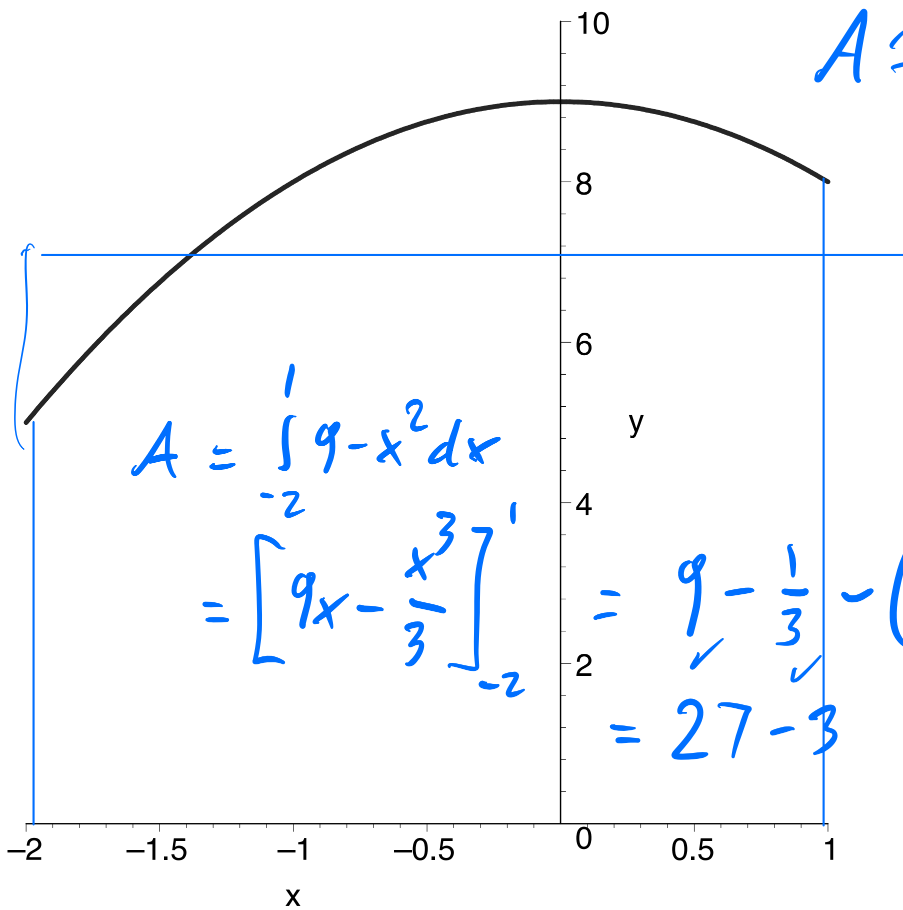


- For example:

$$a = -2, \qquad b = 1, \qquad f(x) = 9 - x^2$$

gives rise to the region shown in the F figure

$A \approx 3 \cdot 7 = 21$



$A = \int_{-2}^{1} 9 - x^2 dx$

$= \left[9x - \frac{x^3}{3} \right]_{-2}^{1}$

$= 9 - \frac{1}{3} - \left(-18 + \frac{8}{3} \right)$

$= 27 - 3 = 24$

Figure 1. $f(x) = 9 - x^2$, $[a, b] = [-2, 1]$.

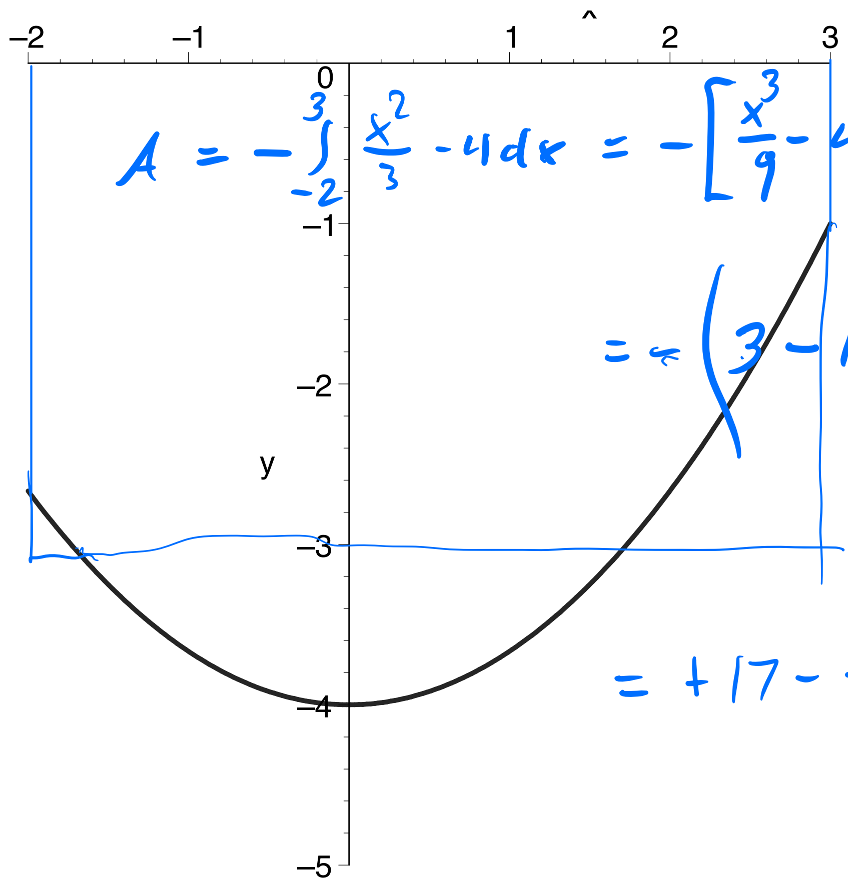
and the integral

$$A = \int_{-2}^{1} 9 - x^2 \mathrm{d}x = 9x - \frac{x^3}{3} \bigg|_{-2}^{1} = 9 - \frac{1}{3} - \left(-18 + \frac{8}{3} \right) = 24.$$

- Example 2 textbook.

$$y = \frac{x^2}{3} - 4, \quad -2 \le x \le 3.$$

$A \approx 5 \cdot 3 = 15$



$A = -\int_{-2}^{3} \frac{x^2}{3} - 4\,dx = -\left[\frac{x^3}{9} - 4x\right]_{-2}^{3}$

$= -\left(3 - 12 - \left(-\frac{8}{9} + 8\right)\right)$

$= +17 - \frac{8}{9} = \frac{153 - 8}{9}$

$= \frac{145}{9}$

Figure 2. Example 2, textbook.

- Example. Compute the area of the region bounded by $x = 0$, $y = 0$, $y = x - 1$, and $x = 3$.

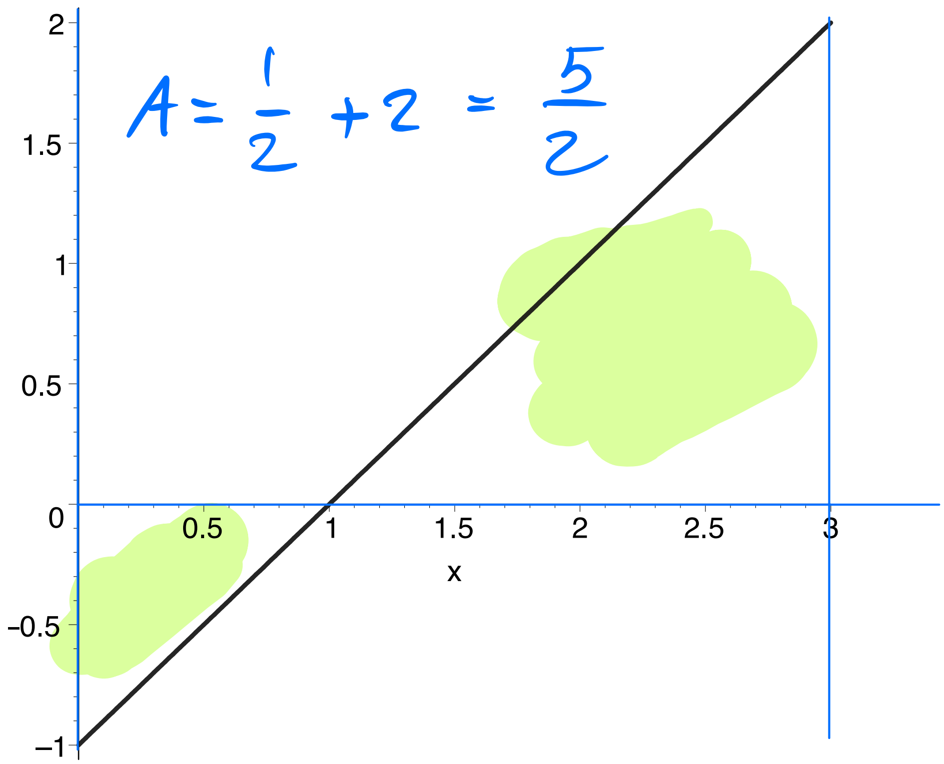


Figure 3. Butterfly.

$$A = -\int_0^1 x - 1\, dx + \int_1^3 x - 1\, dx$$

$$= -\left[\frac{x^2}{2} - x\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^3$$

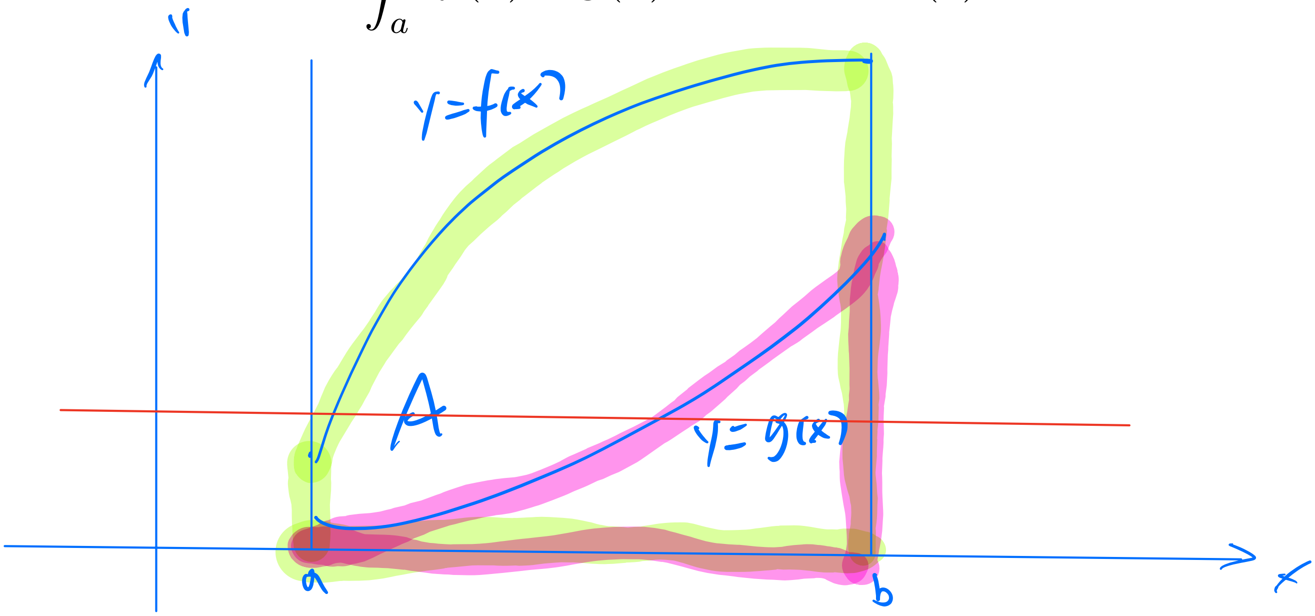
$$= -\left(\frac{1}{2} - 1\right) + \left(\frac{9}{2} - 3 - \left(\frac{1}{2} - 1\right)\right)$$

$$= \frac{1}{2} + (-2 + 4) = \frac{1}{2} + 2 = \frac{5}{2}$$

Region Between Two Curves

- Suppose $f(x) \geq g(x)$ for all x in $[a, b]$. Let A be the area of the region enclosed by the graphs of f and g, and the vertical lines $x = a$ and $x = b$.
- Then

$$A = \int_a^b f(x) - g(x) \mathrm{d}x. \tag{1}$$

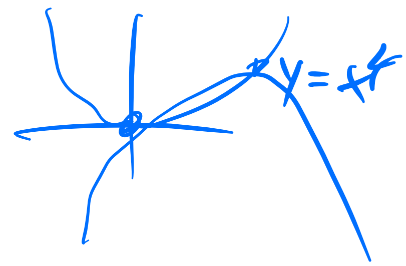


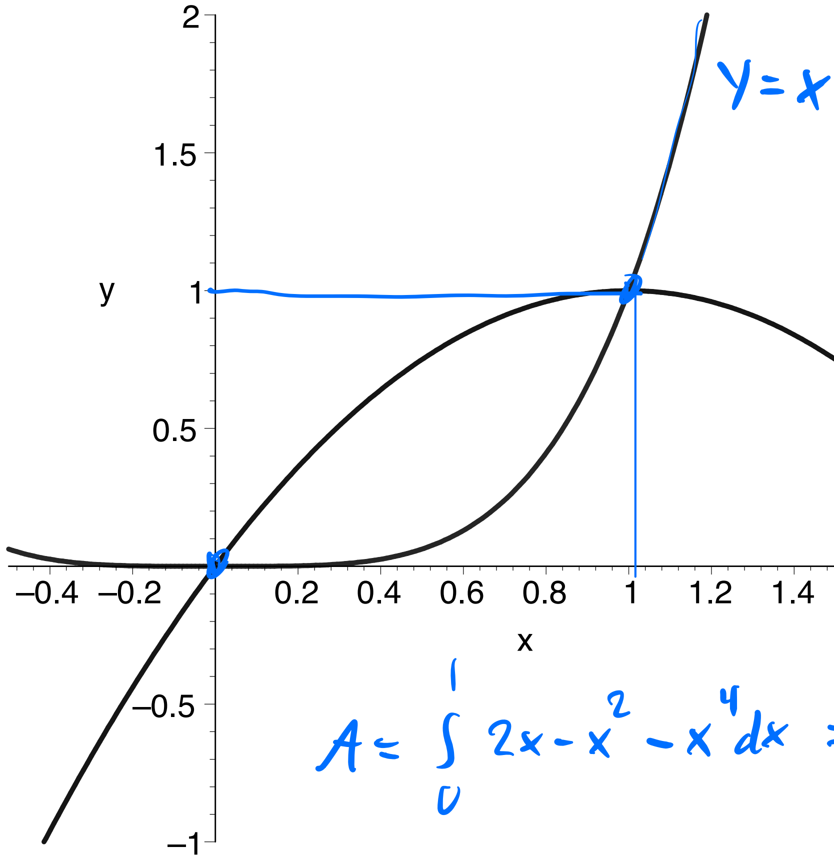
$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx$$

$x^4 = 2x - x^2 \rightsquigarrow x = 0,\ x = 1$

$x^4 + x^2 - 2x = x(x^3 + x - 2)$

- Example 5 textbook. Compute the area of the region enclosed by the curves $y = x^4$ and $y = 2x - x^2$.
- We have to compute the intersections, and the vertical line segments in this case have length zero.





$A \approx \frac{1}{3}\quad \frac{1}{2}$

$A = \int_0^1 2x - x^2 - x^4\,dx = \left[x^2 - \frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$

$= 1 - \frac{1}{3} - \frac{1}{5} = \frac{15-5-3}{15}$

$= \frac{7}{15}$

Figure 4. Example 5.

$4x = y^2$ $\quad y^2 - 3y - 4 = 0 \quad y = -1 \quad x = \frac{1}{4}$

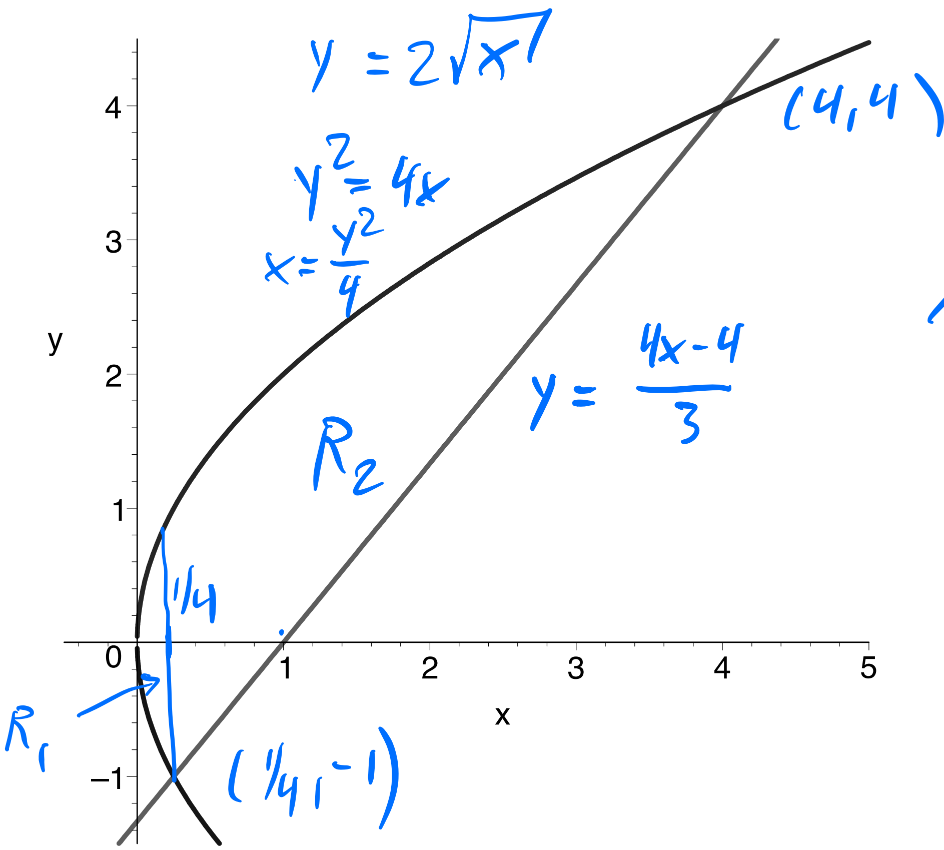
$4x = 3y + 4 \quad (y+1)(y-4) = 0 \quad y = 4 \quad x = 4$

$y^2 = 3y + 4 \quad y = \frac{4x-4}{3} \quad x = \frac{3y+4}{4}$

$(\frac{1}{4}, -1)$

$(4, 4)$

- The formula (1) works even if the x-axis passes through the region of interest.
- You can think about it in terms of Riemann Sums, or just slide both graphs up by the same amount.
- Example 6. Compute the are of the region between $y^2 = 4x$ and $4x - 3y = 4$.



$A \approx 5$

Figure 5. Example 6.

- Breaking the interval $[0,4]$ into the two subintervals $\left[0, \frac{1}{4}\right]$ and $\left[\frac{1}{4}, 4\right]$, and the region of interest into two corresponding regions R_1 and R_2, we get:

$$
\begin{aligned}
A &= A(R_1) + A(R_2) \\
&= \int_0^{\frac{1}{4}} 2\sqrt{x} - (-2\sqrt{x})\mathrm{d}x + \int_{\frac{1}{4}}^{4} 2\sqrt{x} - \frac{4x-4}{3}\mathrm{d}x \\
&= \int_0^{\frac{1}{4}} 4x^{\frac{1}{2}}\mathrm{d}x + \int_{\frac{1}{4}}^{4} 2x^{\frac{1}{2}} - \frac{4x}{3} + \frac{4}{3}\mathrm{d}x \\
&= \left[4 \times \frac{2}{3}x^{\frac{3}{2}}\right]_0^{\frac{1}{4}} + \left[2 \times \frac{2}{3}x^{\frac{3}{2}} - \frac{2x^2}{3} + \frac{4x}{3}\right]_{\frac{1}{4}}^{4} \\
&= \frac{8}{3} \times \frac{1}{4^{\frac{3}{2}}} + \frac{4}{3} \times 4^{\frac{3}{2}} - \frac{32}{3} + \frac{16}{3} - \left(\frac{4}{3} \times \frac{1}{4^{\frac{3}{2}}} - \frac{2}{48} + \frac{1}{3}\right) \\
&\qquad \ldots \text{ noting that } 4^{\frac{3}{2}} = 8 \\
&= \frac{1}{3} + \frac{32}{3} - \frac{16}{3} - \frac{1}{6} + \frac{1}{24} - \frac{1}{3} \\
&= \frac{8 + 256 - 128 - 4 + 1 - 8}{24} \\
&= \frac{125}{24}
\end{aligned}
$$

- There is a much simpler way! Can you think of it?

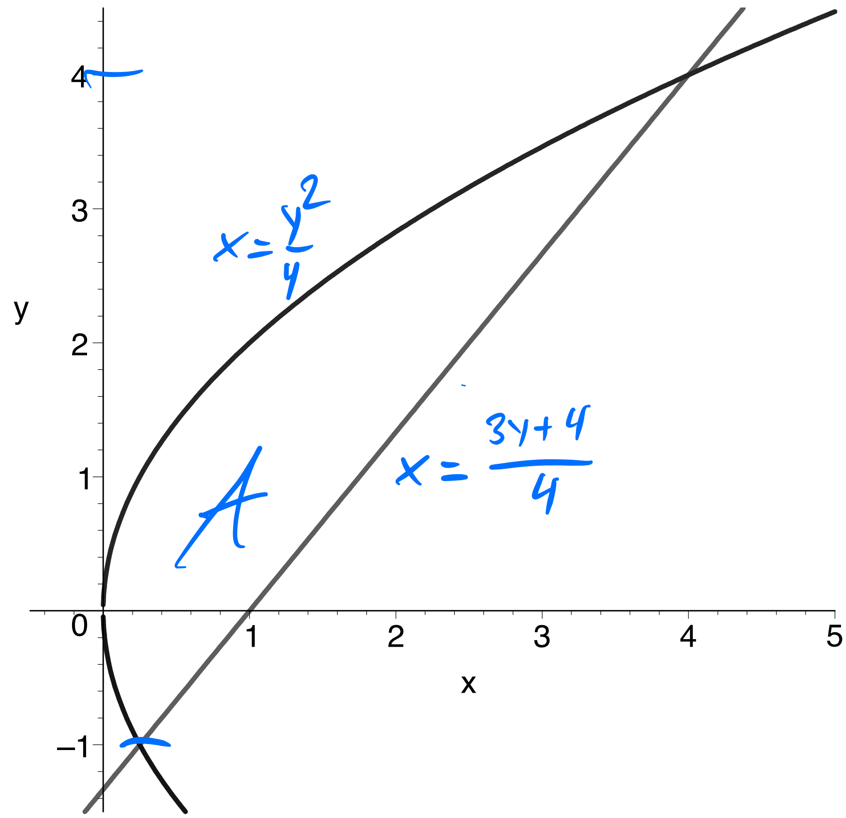


Figure 6. Example 6.

$$A=\int_{-1}^{4}\frac{3y+4}{4}-\frac{y^2}{4}\,dy$$

$$=\int_{-1}^{4}\frac{3y}{4}+1-\frac{y^2}{4}\,dy$$

$$=\left[\frac{3y^2}{8}+y-\frac{y^3}{12}\right]_{-1}^{4}$$

.... ex.

$$=\frac{125}{24}$$

- Example 7: An object is at position $s = 3$ at time $t = 0$. Its velocity at time t is

$$v(t) = 5\sin(6\pi t).$$

What is the object's location at time $t = 2$, and what total distance did it travel during that time?

$s = s(t)$

$v = s'$

$s(2) = ?$

$$s(x) = 3 + \int_0^x v(t)\,dt$$

$s(2) = 3$