Pick up your Exam

- 2 piles of exams, sorted by last name.
- Please keep piles in order for those who come after you.

\[ \text{A—L} \quad \longleftrightarrow \quad \text{M—Z} \]
Definite Integrals—Quick Review

- Recall that $\int_a^b f(x)\,dx$ is the definite integral of $f$ with respect to $x$ from $a$ to $b$.
- $x$ is the integration variable.
- $f(x)$ is the integrand.
- $a$ and $b$ are the lower and upper limits of integration, respectively.
- We defined the definite integral as the limit of a Riemann Sum.
- Geometrically the definite integral gives the area of the region under the curve.
Notation

• Here are some frequent notations for the definite integral.

• Suppose $F$ is an antiderivative of $f$, i.e.,

$$F' = f.$$ 

$$\int_a^b f(x) \, dx = F(b) - F(a)$$ 

$$= \left[ F(x) \right]_a^b$$ 

$$= \left[ F(x) \right]_{x=a}^{x=b}$$ 

$$= F(b) \bigg|_a$$ 

$$= F(b) \bigg|_{x=a}$$
4.3-4.4 The Fundamental Theorem of Calculus

- We’ll spend two days on sections 4.3 and 4.4 combined.

- Recall that we introduced derivatives and integrals by going back and forth between velocity and location.

- Naturally these two processes are inverses of each other.

- The Fundamental Theorem of Calculus (FToC) makes this precise.

- It comes in two flavors.

- Theorem A, page 235

\[
\frac{d}{dx} \int_a^x f(t) dt = f(x)
\]

- Theorem A, page 243

\[
\int_a^b f(x) dx = F(b) - F(a) \quad \text{where} \quad F'(x) = f(x).
\]
• The textbook calls these the first and second fundamental theorem of Calculus, but the two statements are equivalent and there is really only one FToC.

• We need to learn how to use these facts, and we need to see why they are true and why they are equivalent.

• Before thinking about why the statements are true and equivalent, let’s do some examples.

• Example:
\[
\frac{d}{dx} \int_0^x t^2 \, dt = x^2
\]

• Example:
\[
\int_0^2 x^2 \, dx = \left[ \frac{x^3}{3} \right]_0^2 = \frac{8}{3} - 0 = \frac{8}{3}
\]

• Example:
\[
\int_0^\pi \sin t \, dt = -\cos t \bigg|_0^\pi = 1 + 1 = 2
\]

• Example:
\[
\int_1^3 3x + 4 \, dx = \frac{3x^2}{2} + 4x \bigg|_1^3 = \frac{27}{2} + 12 - \left( \frac{3}{2} + 4 \right) = 20
\]

• Example:
\[
\int_3^5 4x^3 + 1 \, dx = x^4 + x \bigg|_3^5 = 625 + 5 - (81 + 3)
\]
\[ \frac{d}{dx} \int_0^{x^2} f(t) \, dt = f(x^2) \]

\[ \frac{d}{dx} \int_0^x f(t) \, dt = f(x) \]

\[ \frac{d}{dx} \int_0^{x^2} f(t) \, dt = \frac{d}{dx} \left( F(x^2) + F(0) \right) \]

\[ = F'(x^2) \cdot 2x \]

\[ = f(x^2) \cdot 2x \]

\[ = 2x \cdot f(x^2) \]
Why is the FToC true?

\[ \frac{d}{dx} \int_{a}^{x} f(t) \, dt = f(x) \]

\[ A(x) = \int_{a}^{x} f(t) \, dt \]

\[ \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = f(x) \]
• The second version follows easily from the first.

\[
\frac{d}{dx} \int_a^b f(t) \, dt = f(x)
\]

\[
\Rightarrow A(b) - \int_a^b f(t) \, dt = F(b) - F(a)
\]

\[
A' = f
\]

\[
F(x) = A(x) + C
\]

\[
F(b) - F(a) = A(b) + C - (A(a) + C) = A(b) + C - A(a) - C = 0
\]
• The first version follows easily from the second.

\[
\frac{d}{dx} \int_{a}^{x} f(t) \, dt = \frac{d}{dx} (F(x) - F(a))
\]

\[
= f(x) - 0
\]

\[
= f(x)
\]
• The two versions of the FToC are two sides of the same coin.

• There is only one FToC.

• Integration and Differentiation are opposite processes.

• Another Example.

• Suppose

\[ G(x) = \int_0^x 2t + 3t^2 \, dt. \]

• Compute \( G'(x) \) in two different ways.

\[
\begin{align*}
\int_0^x 2t + 3t^2 \, dt &= \left. t^2 + \frac{3}{3} t^3 \right|_0^x \\
&= (x^2)^2 + (x^2)^3 = x^4 + x^6 \\
G'(x) &= \frac{d}{dx} (x^4 + x^6) = 4x^3 + 6x^5
\end{align*}
\]
\[
\frac{d}{dx} \int_0^x t^2 \, dt = 2xe^x
\]

\[
\int_0^{\pi/2} \sin t \cos t \, dt = \frac{1}{2} \sin^2 \frac{\pi}{2} = \frac{1}{2}
\]

\[
\frac{d}{dx} \int_0^{\sin x} \cos t \, dt = \cos(\sin x) \cos x
\]

\[
\frac{d}{dt} (-\sin t \cos t) = -\cos^2 t + \sin^2 t = -
\]

\[
\frac{d}{dx} \int_{U(x)}^{L(x)} f(t) \, dt = \frac{d}{dx} \left( F(L(x)) - F(U(x)) \right)
\]

\[
= F' \left( U(x) \right) U'(x) - F' \left( L(x) \right) L'(x)
\]

\[
= f(U(x)) \cdot U'(x) - f(L(x)) \cdot L'(x)
\]