Math 1210-23

Notes of 3/29/24

5.1 The Area of a Plane Region

- We saw that if $f(x) \geq 0$ for all x in $[a, b]$ then

  $$A = \int_a^b f(x)\mathrm{d}x$$

  is the area of the region enclosed by the x-axis, the lines $x = a$ and $x = b$, and the graph of f.

- For example:

$$a = -2, \qquad b = 1, \qquad f(x) = 9 - x^2$$

gives rise to the region shown in the F figure

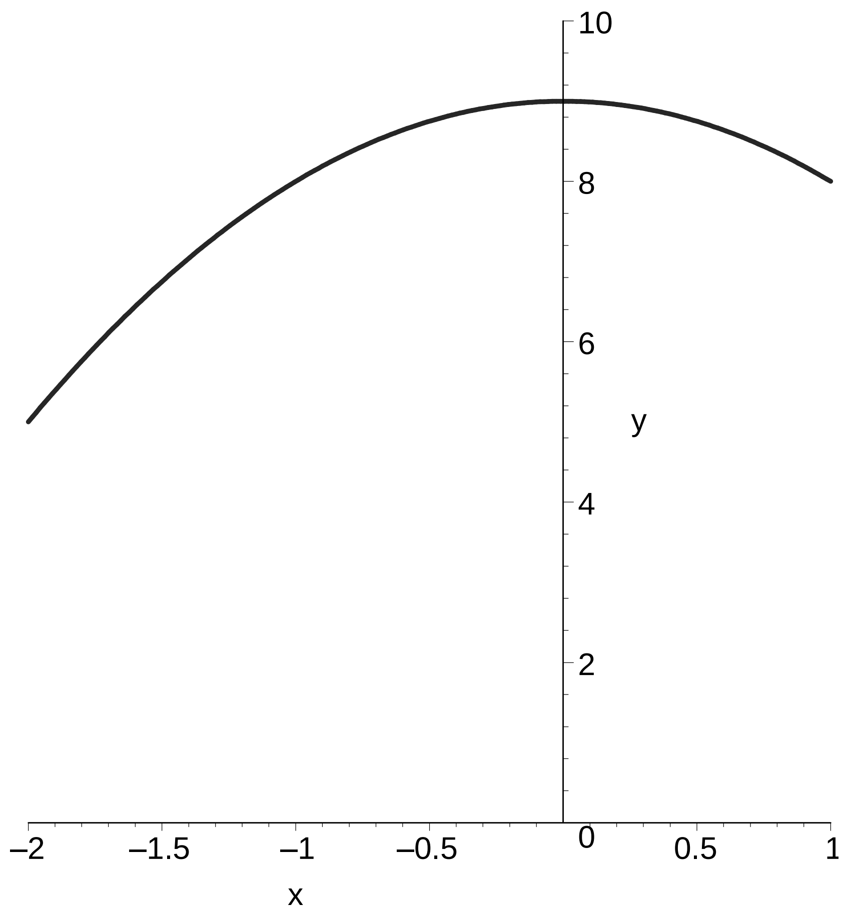

Figure 1. $f(x) = 9 - x^2$, $[a, b] = [-2, 1]$.

and the integral

$$A = \int_{-2}^{1} 9 - x^2 \mathrm{d}x = 9x - \frac{x^3}{3}\bigg|_{-2}^{1} = 9 - \frac{1}{3} - \left(-18 + \frac{8}{3}\right) = 24.$$

- Example 2 textbook.

$$y = \frac{x^2}{3} - 4, \quad -2 \le x \le 3.$$

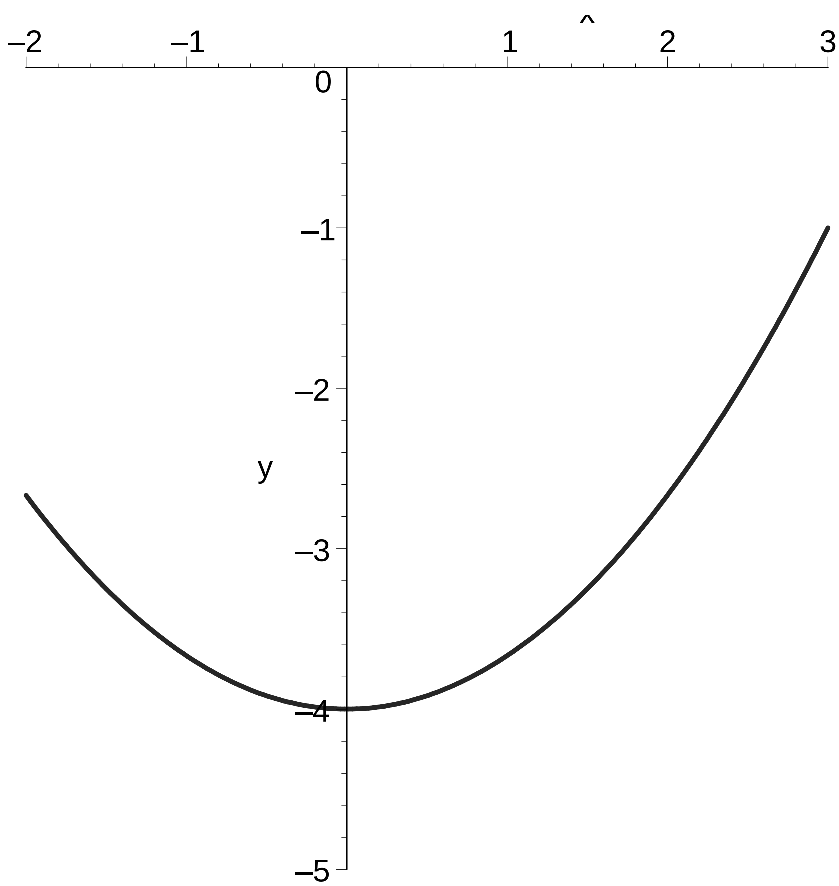

Figure 2. Example 2, textbook.

- Example. Compute the area of the region bounded by $x = 0$, $y = 0$, $y = x - 1$, and $x = 3$.

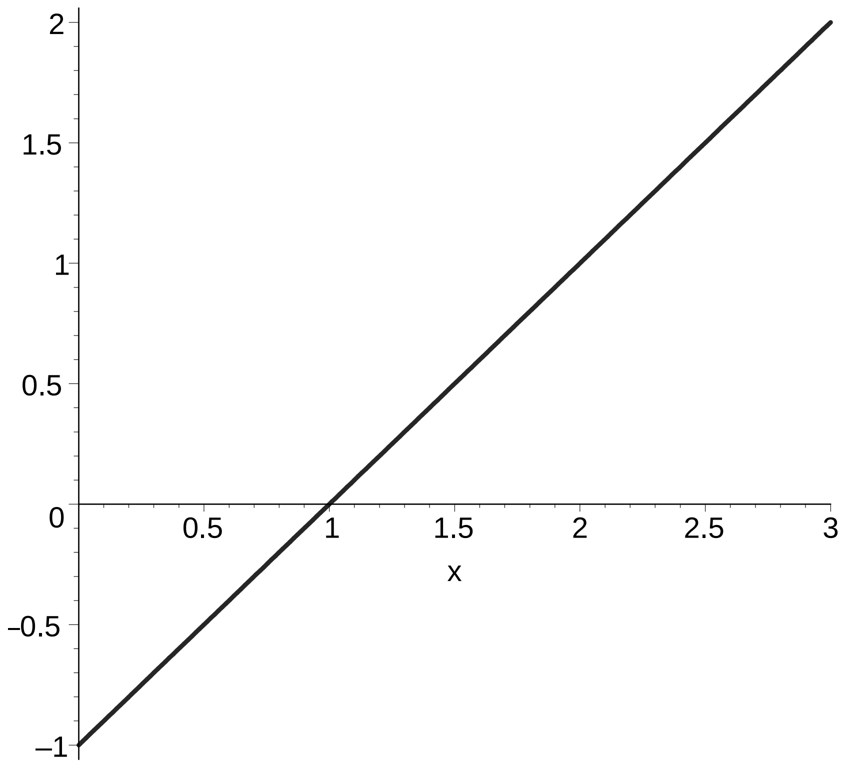

Figure 3. Butterfly.

Region Between Two Curves

- Suppose $f(x) \geq g(x)$ for all x in $[a, b]$. Let A be the area of the region enclosed by the graphs of f and g, and the vertical lines $x = a$ and $x = b$.
- Then

$$A = \int_a^b f(x) - g(x) \mathrm{d}x. \tag{1}$$

- Example 5 textbook. Compute the area of the region enclosed by the curves $y = x^4$ and $y = 2x - x^2$.
- We have to compute the intersections, and the vertical line segments in this case have length zero.

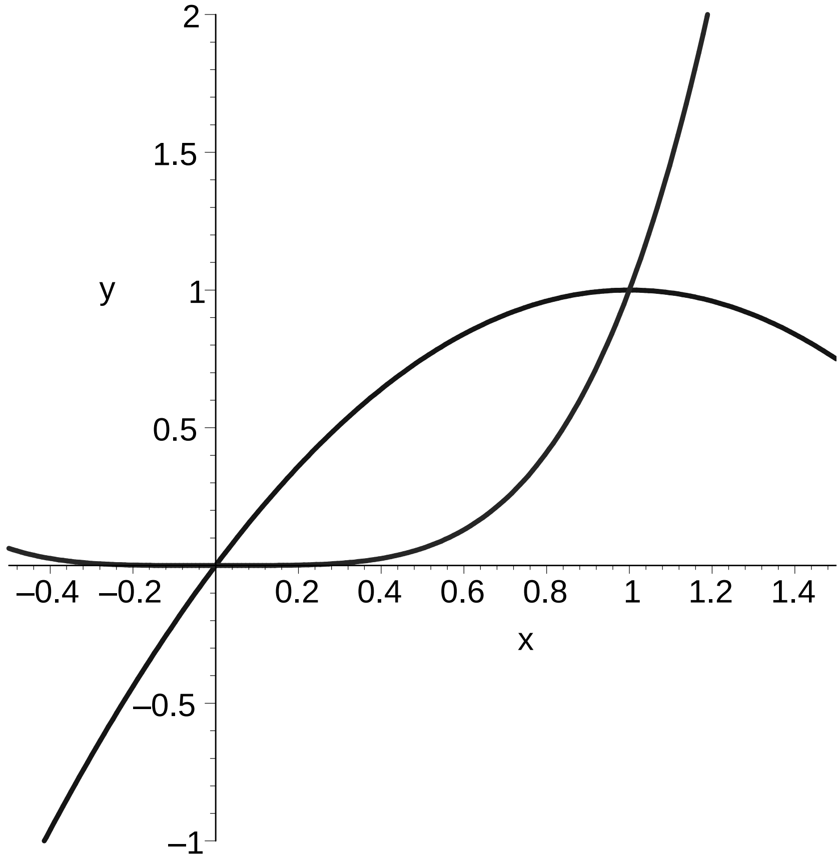

Figure 4. Example 5.

- The formula (1) works even if the x-axis passes through the region of interest.
- You can think about it in terms of Riemann Sums, or just slide both graphs up by the same amount.
- Example 6. Compute the are of the region between $y^2 = 4x$ and $4x - 3y = 4$.

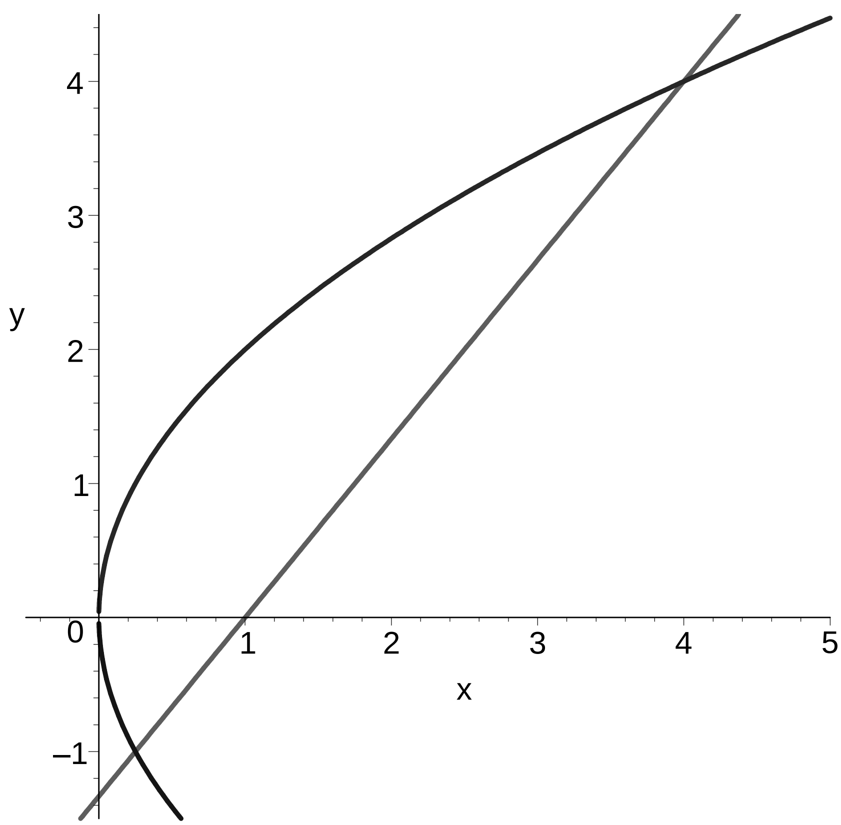

Figure 5. Example 6.

- Breaking the interval $[0,4]$ into the two subintervals $\left[0,\frac{1}{4}\right]$ and $\left[\frac{1}{4},4\right]$, and the region of interest into two corresponding regions R_1 and R_2, we get:

$$\begin{aligned}
A &= A(R_1) + A(R_2) \\
&= \int_0^{\frac{1}{4}} 2\sqrt{x} - (-2\sqrt{x})\mathrm{d}x + \int_{\frac{1}{4}}^4 2\sqrt{x} - \frac{4x-4}{3}\mathrm{d}x \\
&= \int_0^{\frac{1}{4}} 4x^{\frac{1}{2}}\mathrm{d}x + \int_{\frac{1}{4}}^4 2x^{\frac{1}{2}} - \frac{4x}{3} + \frac{4}{3}\mathrm{d}x \\
&= \left[4 \times \frac{2}{3}x^{\frac{3}{2}}\right]_0^{\frac{1}{4}} + \left[2 \times \frac{2}{3}x^{\frac{3}{2}} - \frac{2x^2}{3} + \frac{4x}{3}\right]_{\frac{1}{4}}^4 \\
&= \frac{8}{3} \times \frac{1}{4^{\frac{3}{2}}} + \frac{4}{3} \times 4^{\frac{3}{2}} - \frac{32}{3} + \frac{16}{3} - \left(\frac{4}{3} \times \frac{1}{4^{\frac{3}{2}}} - \frac{2}{48} + \frac{1}{3}\right) \\
&\qquad \ldots \text{ noting that } 4^{\frac{3}{2}} = 8 \\
&= \frac{1}{3} + \frac{32}{3} - \frac{16}{3} - \frac{1}{6} + \frac{1}{24} - \frac{1}{3} \\
&= \frac{8 + 256 - 128 - 4 + 1 - 8}{24} \\
&= \frac{125}{24}
\end{aligned}$$

- There is a much simpler way! Can you think of it?

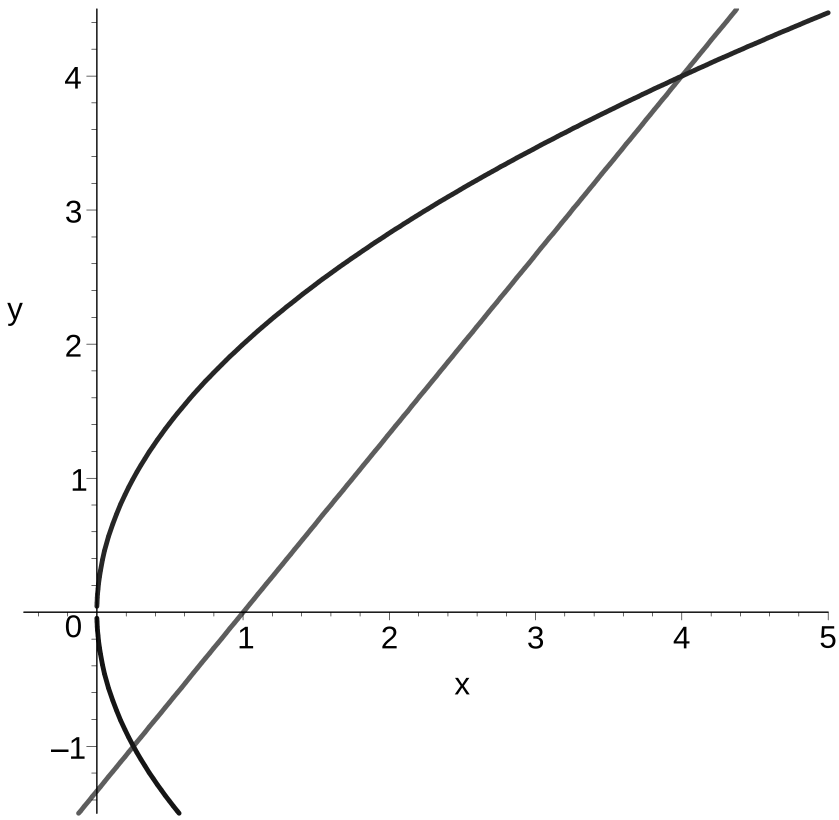

Figure 6. Example 6.

- Example 7: An object is at position $s = 3$ at time $t = 0$. Its velocity at time t is

$$v(t) = 5\sin(6\pi t).$$

  What is the object's location at time $t = 2$, and what total distance did it travel during that time?