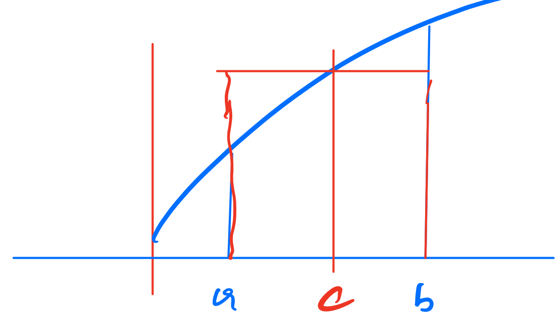
Notes of 3/29/22



The Mean Value Theorem for Integrals

- The MVT for Integrals says that if f is continuous on $[a, b]$ there must be a point c in (a, b) such that f at that point equals the average value.
- That seems geometrically obvious.
- Stated more formally we have:
- Suppose f is continuous on $[a, b]$. Then there exists a number c in (a, b) such that

$$f(c) = \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t.$$

- This can be rewritten as

$$F'(c)(b-a) = f(c)(b-a) = \int_a^b f(t)\mathrm{d}t = F(b)-F(a)$$

- Note that in particular,

$$f(c)(b-a) = F(b) - F(a)$$

is just the mean value value theorem for derivatives applied to the function F.

- Example: Compute c for $f(x) = x^p$ on the interval $[0, 1]$.

$f(x) = x^p$

$$f(x) = x^p$$

$$\int_0^1 x^p dx = \left. \frac{x^{p+1}}{p+1} \right|_0^1 = \frac{1}{p+1}$$

$$f(c) = c^p \underbrace{(1-0)}_{1} = \frac{1}{p+1}$$

$$f(c) = c^p = \frac{1}{p+1}$$

$$c = \left(\frac{1}{p+1}\right)^{1/p} = c(p)$$

- The following Table shows c for some values of $p > 0$:

$p:$	1	2	3	4	5	6	7	8	9	10	1,000
$c:$	0.5	0.58	0.63	0.67	0.70	0.72	0.74	0.76	0.77	0.79	0.99

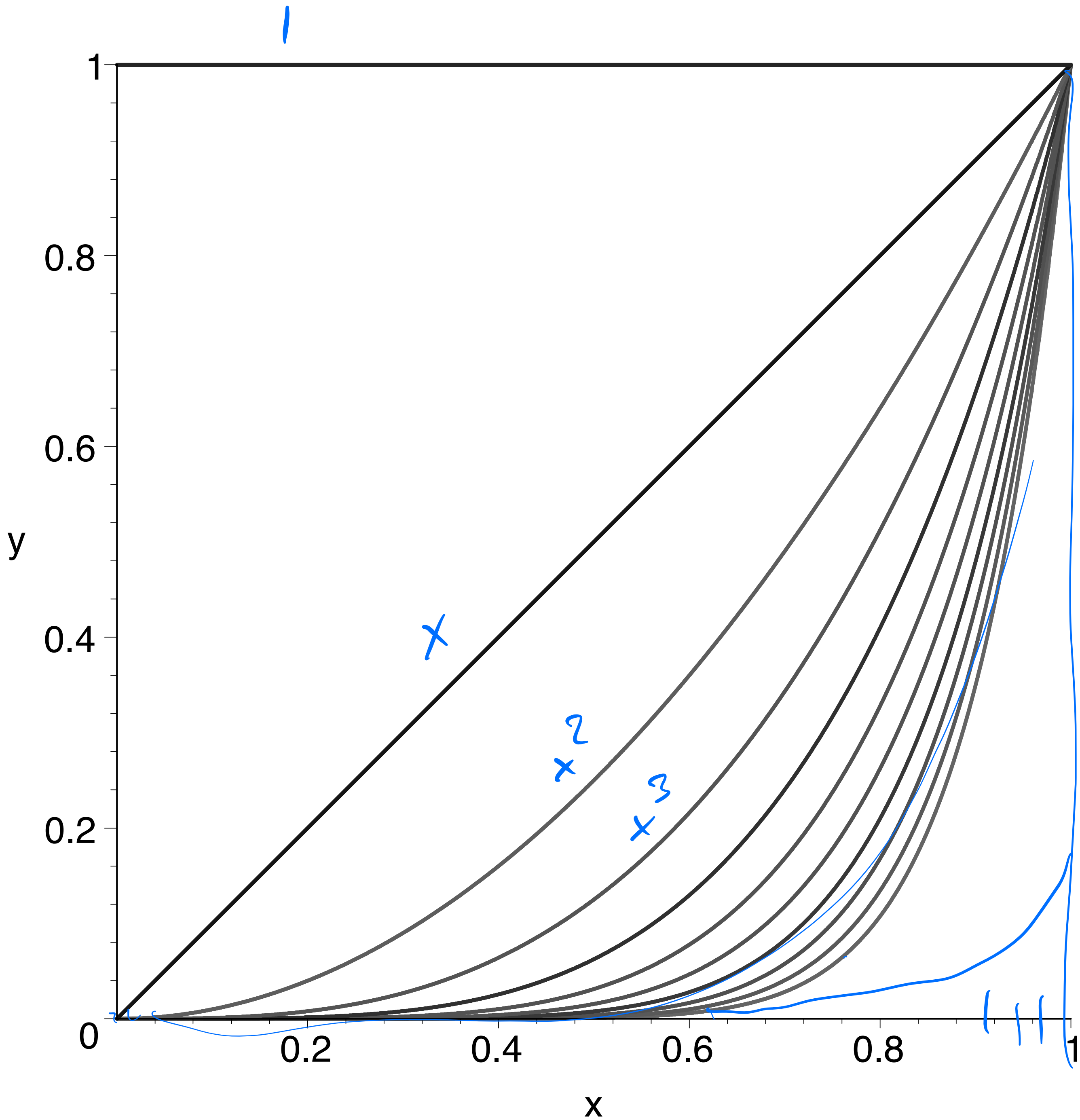


Figure 1. $f(x) = x^p, \quad p = 0, \ldots, 10.$

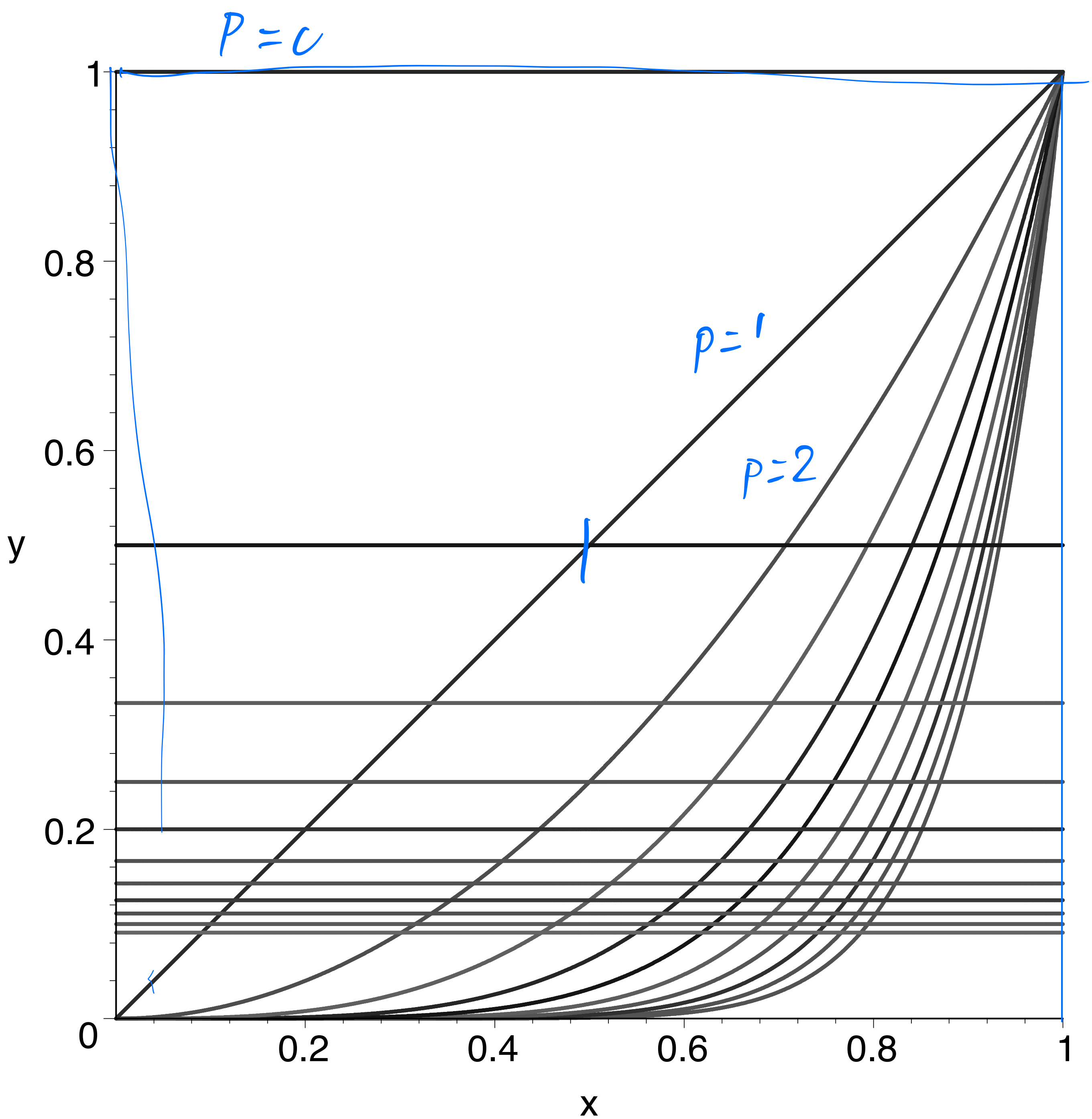
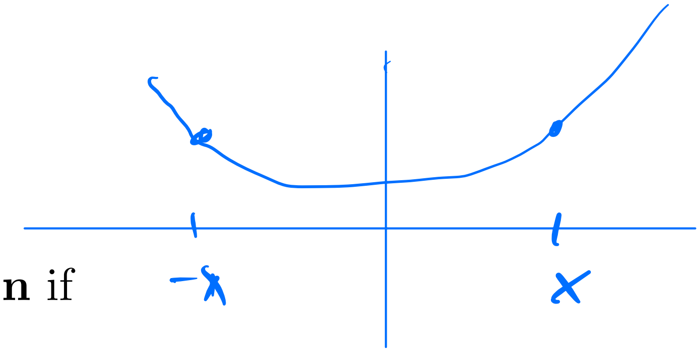
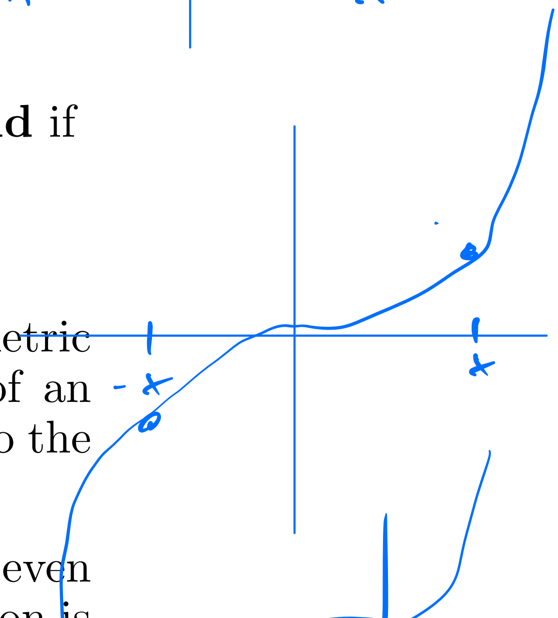


Figure 2. $f(x) = x^p, \quad p = 0, \ldots, 10$ and average values.

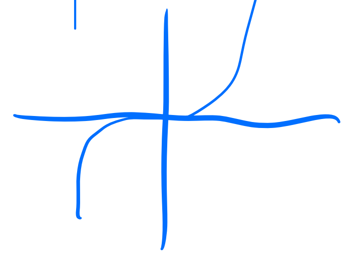
Symmetry



- Recall that a function f is **even** if
$$f(x) = f(-x)$$
for all x in the domain of f, and it is **odd** if
$$f(x) = -f(-x)$$
for all x in the domain of f.

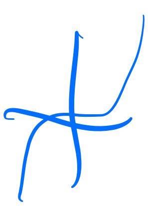


- The graph of an even function is symmetric with respect to the y-axis, the graph of an odd function is symmetric with respect to the origin.



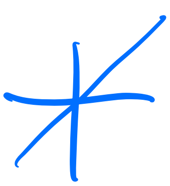
$f(x)=x^2$

- It's easy to see that the derivative of an even function is odd and that of an odd function is even. Suppose f is even. Then we get



$y=x^3$

$$\begin{array}{rclr} f(x) & = & f(-x) & f \text{ is even} \\ f'(x) & = & -f'(-x) & f' \text{ is odd} \\ f''(x) & = & f''(-x) & f'' \text{ is even} \end{array}$$

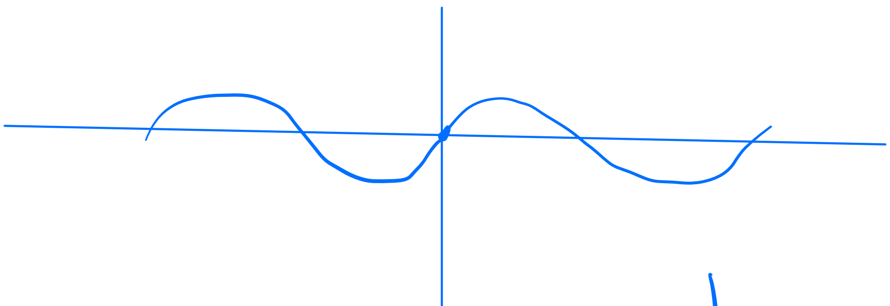


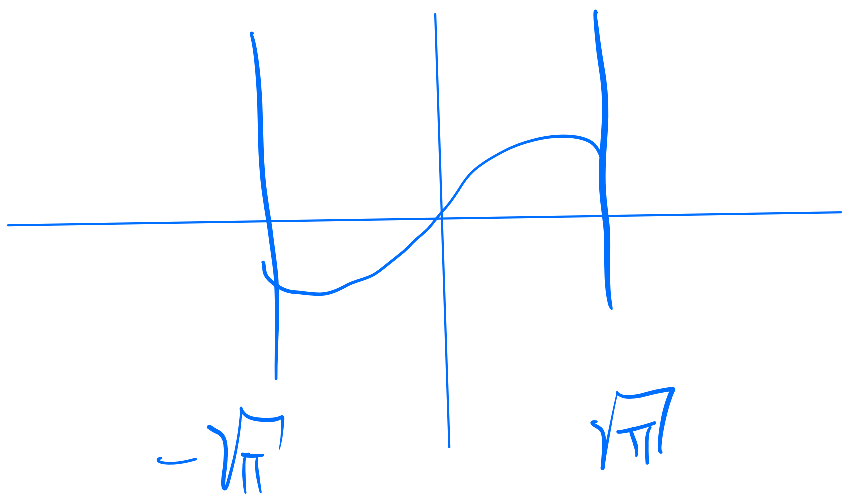
- With the right choice of the integration constant we can also go the other way. (Think about what the word "right" means.)
- Thus we get the fact

$$\boxed{\textbf{The } \begin{matrix}\textbf{derivative}\\ \textbf{antiderivative}\end{matrix} \textbf{ of an } \begin{matrix}\textbf{even}\\ \textbf{odd}\end{matrix} \textbf{ function is } \begin{matrix}\textbf{odd}\\ \textbf{even}\end{matrix}.}$$

- A little **puzzle** for you: Compute

$$\int_{-\sqrt{\pi}}^{\sqrt{\pi}} \left(\sin x + \sin^{13} x^{17} \right)^{15} \mathrm{d}x = 0$$





- in general, if f is odd, then

$$\int_{-b}^{0} f(t)\mathrm{d}t = -\int_{0}^{b} f(t)\mathrm{d}t$$

and

$$\int_{-b}^{b} f(t)\mathrm{d}t = 0$$

$$\int_{-\pi}^{\pi} x^{17} = 0$$

$$\int_{-1}^{\pi} x^{17} \neq 0$$

- Examples:

$$\int_{-1}^{1} t^3\mathrm{d}t = \quad 0$$

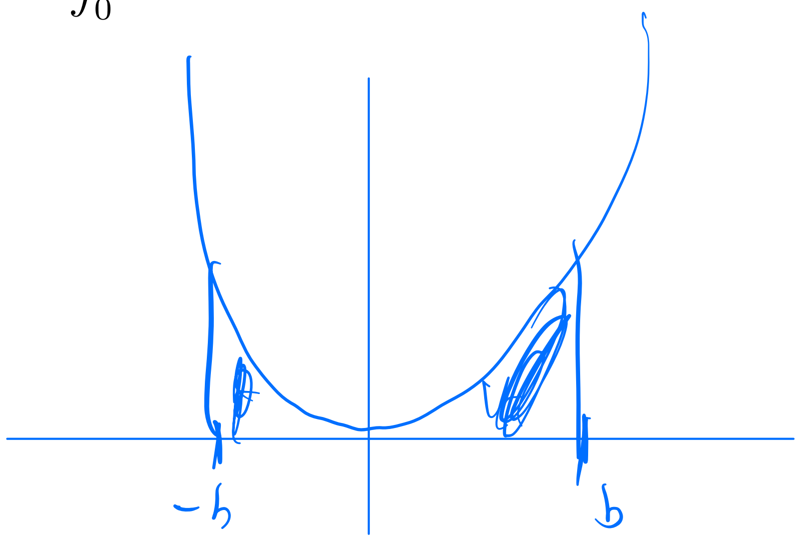
$$\int_{-1}^{1} \sin x^3\mathrm{d}x = \quad 0$$

- The corresponding property for even functions is not nearly as useful:

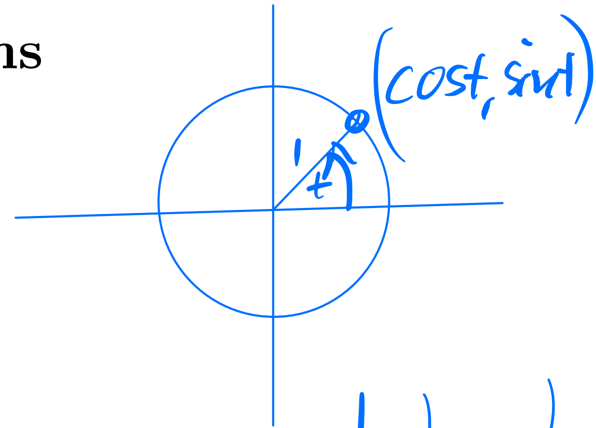
$$\int_{-b}^{0} f(t)\mathrm{d}t = \int_{0}^{b} f(t)\mathrm{d}t$$

and

$$\int_{-b}^{b} f(t)\mathrm{d}t = 2\int_{0}^{b} f(t)\mathrm{d}t.$$



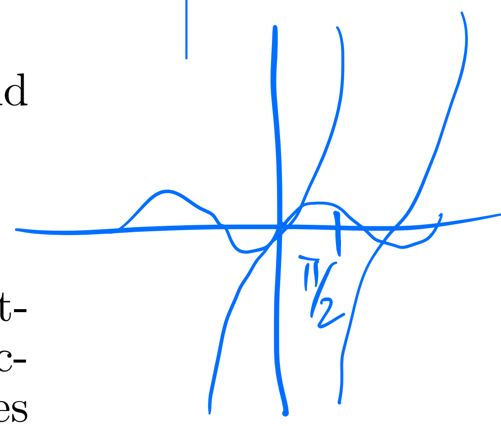
Integrals of periodic functions



- f is p-periodic if

$$f(t+p) = f(t)$$

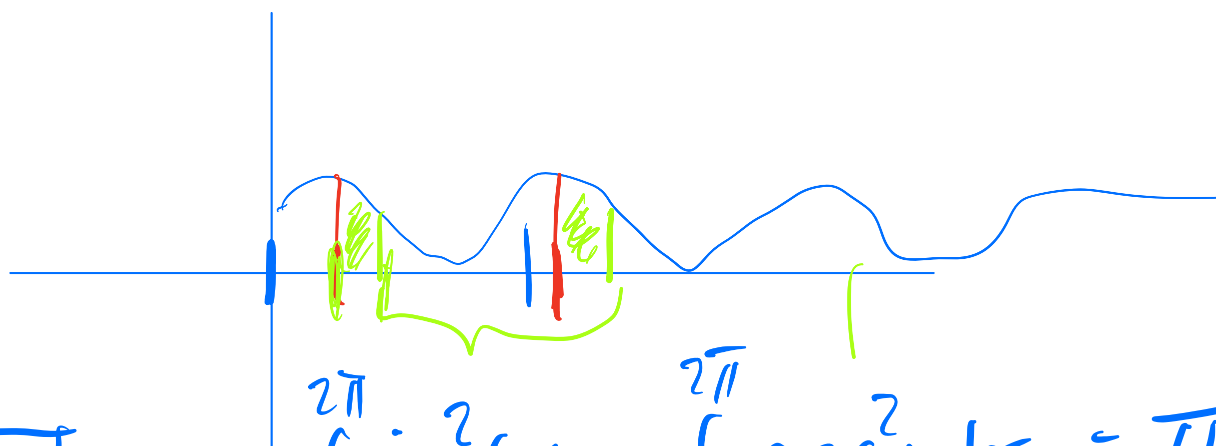
  for all t in the domain of f.

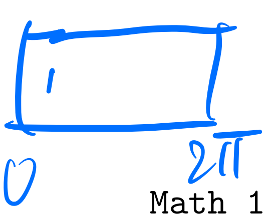


- For example, sin and cos are 2π-periodic (and also, for example, 6π-periodic.
- The tan function is π-periodic.
- Useful property, not mentioned in the textbook: When you integrate a periodic function over an integer number of periods it does not matter where you start. Assuming f is p-periodic, we get

$$\int_a^{a+p} f(t)\mathrm{d}t = \int_{a+z}^{a+p+z} f(t)\mathrm{d}t$$

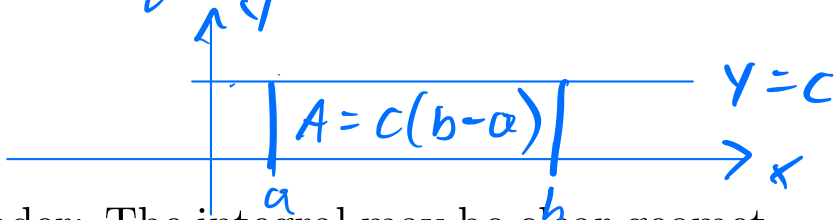
  for all real numbers z.





$$\int_0^{2\pi} \sin^2 dx = \int_0^{2\pi} \cos^2 x\, dx = \pi$$

$$\int^{2\pi} \sin^2 dx + \cos^2 x\, dx = 2\pi$$



- Reminder: The integral may be clear geometrically, even if we are unable to find an antiderivative. One of the most frequently arising examples is

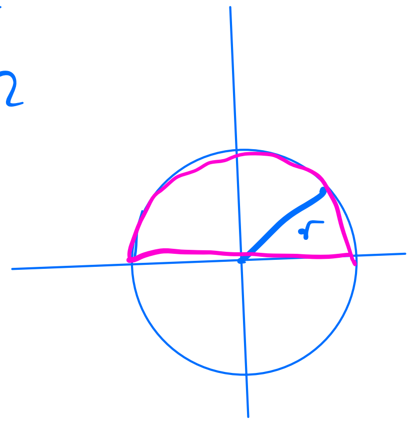
$$\int_a^b c\,dx = cx\Big|_a^b = cb - ca = c(b-a)$$

$$\int_{-r}^{r} \sqrt{r^2 - x^2}\mathrm{d}x = \frac{\pi r^2}{2}$$

$$y = \sqrt{r^2 - x^2}$$

$$y^2 = r^2 - x^2$$

$$x^2 + y^2 = r^2$$



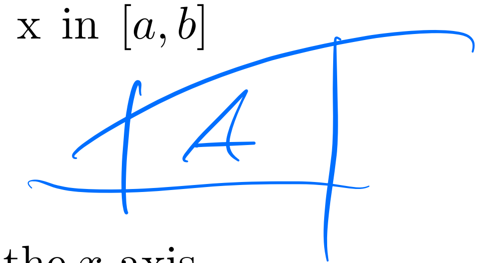
- Note: we don't yet know how to do this (and will learn more in Math 1220) but for your information and entertainment:

$$\int \sqrt{r^2 - x^2}\mathrm{d}x = \frac{1}{2}\left(x\sqrt{r^2 - x^2} + \arctan\frac{x}{\sqrt{r^2 - x^2}}\right) + C.$$

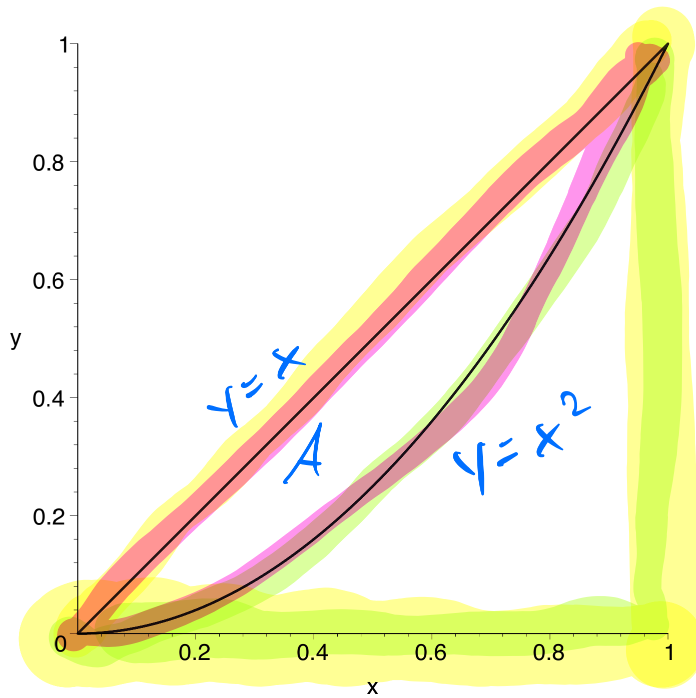
5.1 Computation of Area

- Setting the stage, more on Friday:
- We know that if $f(x) > 0$ for all x in $[a, b]$ then

$$\int_a^b f(x)\mathrm{d}x$$



is the area of the region enclosed by the x-axis, the graph of f, and the vertical lines $x = a$ and $x = b$.



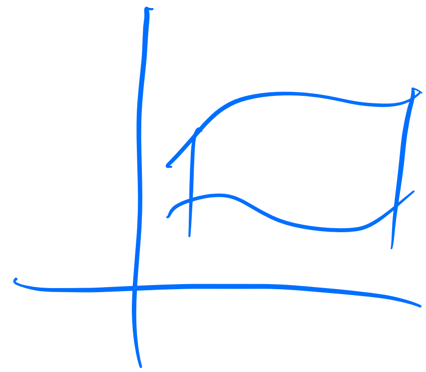


Figure 3. The region enclosed by $\cancel{y}x$ and $y = x^2$.

$y=x$

- What about, for example, the region enclosed by the graphs of $y = x$ and $y = x^2$, as shown in Figure 3?

$$A = \int_0^1 x\,dx - \int_0^1 x^2\,dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$= \int_0^1 x - x^2\,dx = \frac{1}{6}$$

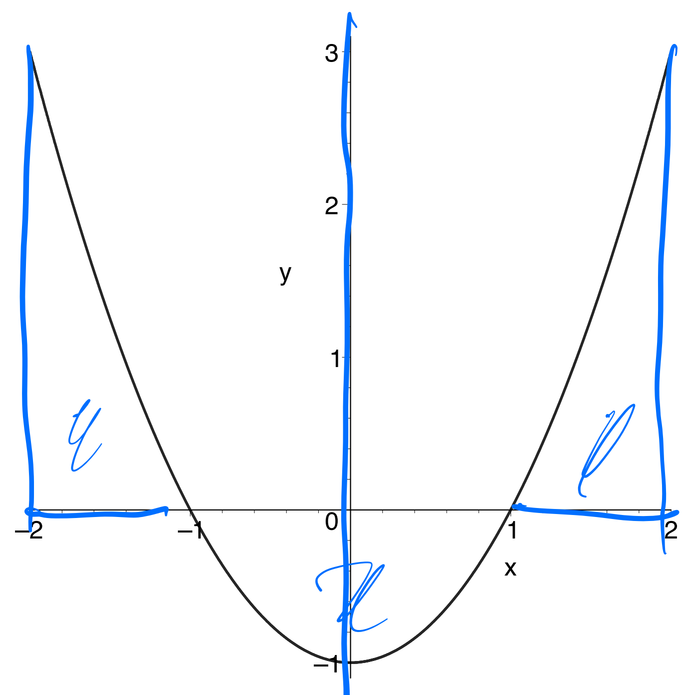


Figure 4. A three part region.

- Or, how about the region enclosed by the graphs of $y = x^2 - 1$, the x-axis, and the vertical lines $x = \pm 2$ shown in Figure 4?

$$A = 2\int_0^1 x^2 - 1\,dx + \int_1^2 x^2 - 1\,dx$$

$$\int_0^1 x^2 - 1 = \frac{x^3}{3} - x \Big|_0^1 = -\frac{2}{3}$$

$$\int_1^2 x^2 - 1\,dx = \frac{x^3}{3} - x \Big|_1^2 = \frac{8}{3} - 2$$

$$A = 2\left(\frac{2}{3} + \frac{8}{3} - 2\right) = \frac{8}{3}$$