Notes of 3/29/22

The Mean Value Theorem for Integrals

- The MVT for Integrals says that if f is continuous on $[a, b]$ there must be a point c in (a, b) such that f at that point equals the average value.
- That seems geometrically obvious.
- Stated more formally we have:
- Suppose f is continuous on $[a, b]$. Then there exists a number c in (a, b) such that

$$f(c) = \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t.$$

- This can be rewritten as

$$F'(c)(b-a) = f(c)(b-a) = \int_a^b f(t)\mathrm{d}t = F(b)-F(a)$$

- Note that in particular,

$$f(c)(b-a) = F(b) - F(a)$$

  is just the mean value value theorem for derivatives applied to the function F.

- Example: Compute c for $f(x) = x^p$ on the interval $[0, 1]$.

- The following Table shows c for some values of $p > 0$:

$p:$	1	2	3	4	5	6	7	8	9	10	1,000
$c:$	0.5	0.58	0.63	0.67	0.70	0.72	0.74	0.76	0.77	0.79	0.99

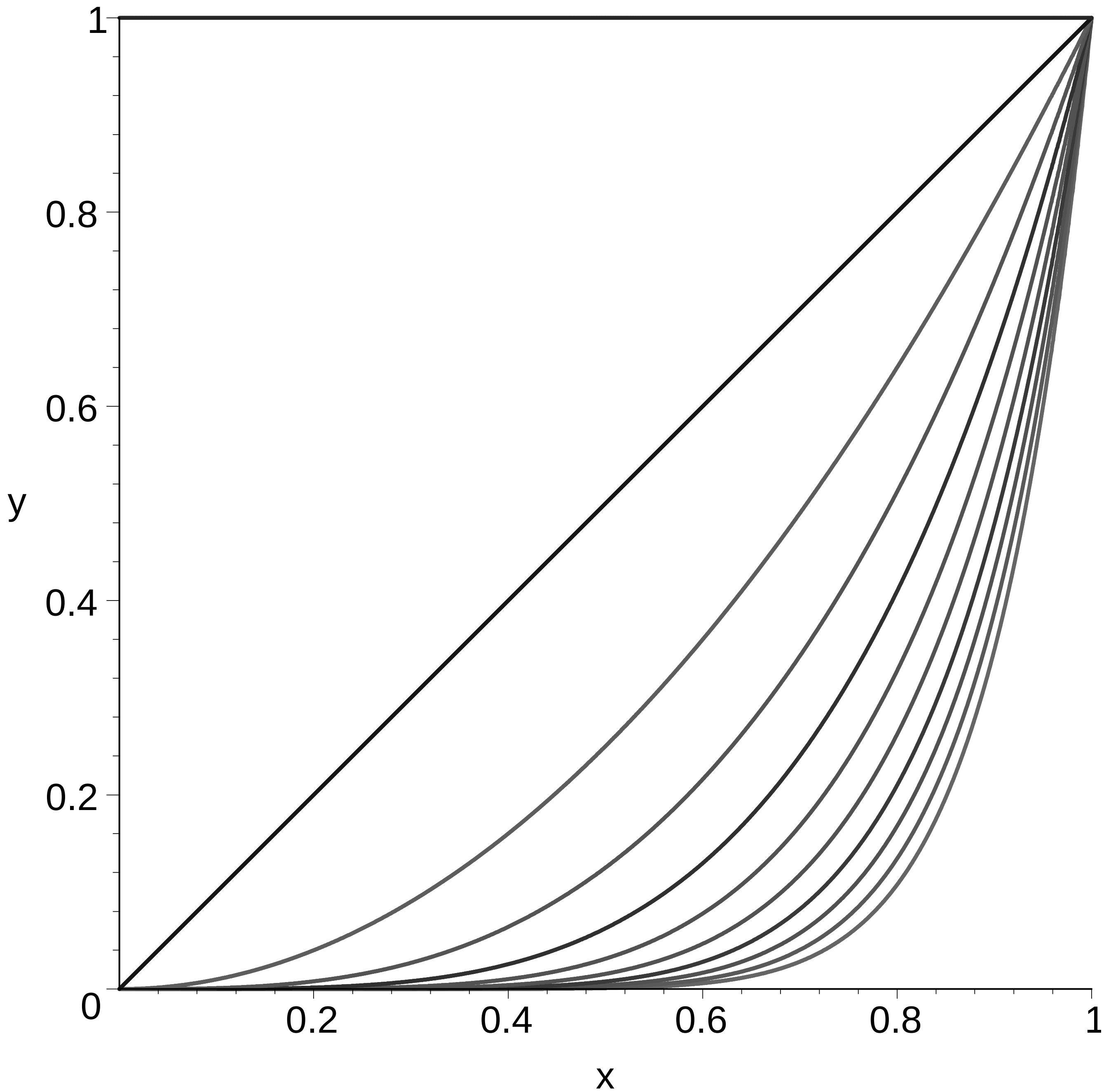

Figure 1. $f(x) = x^p, \quad p = 0, \ldots, 10.$

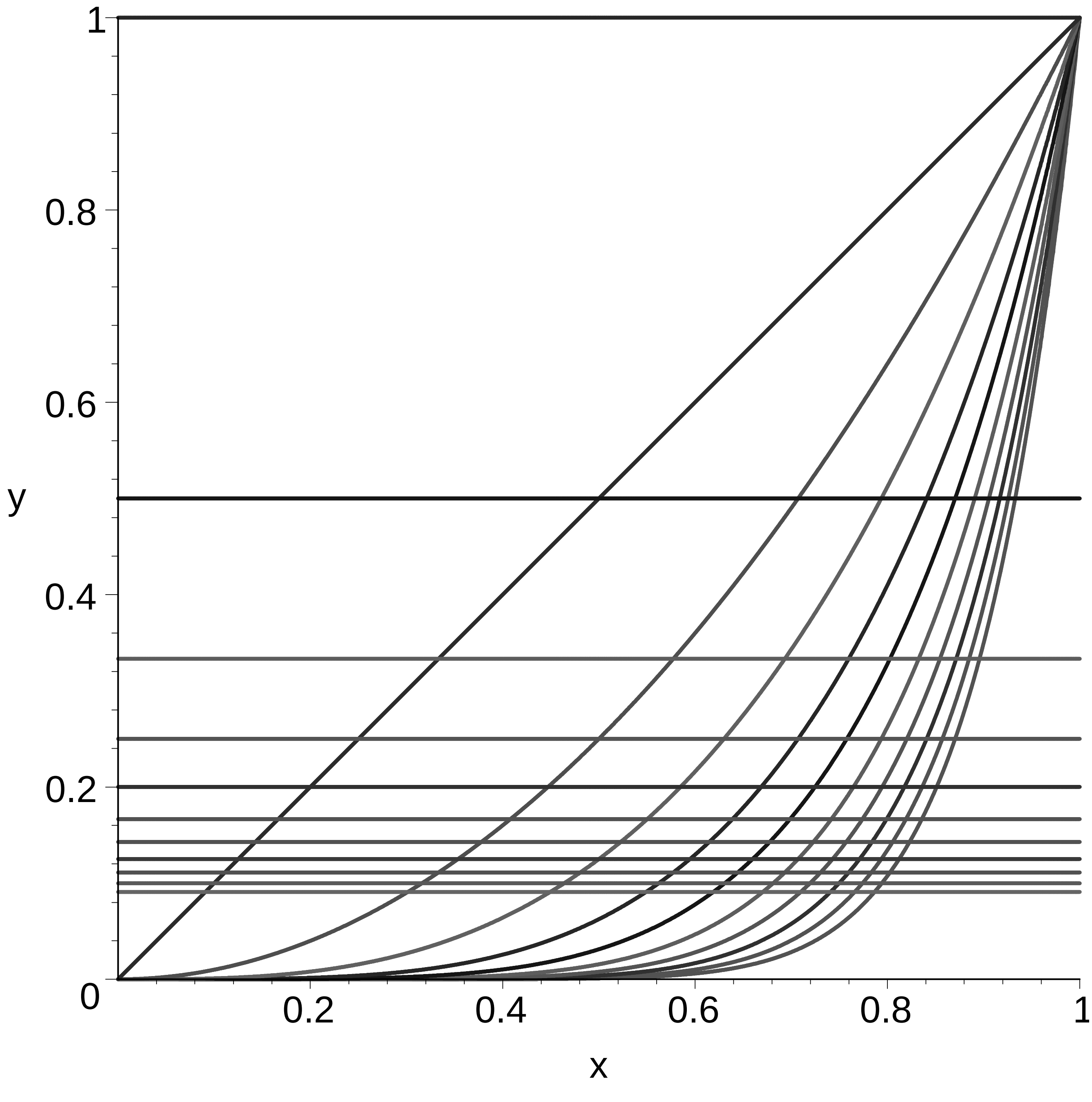

Figure 2. $f(x) = x^p, \quad p = 0, \ldots, 10$ and average values.

Symmetry

- Recall that a function f is **even** if
$$f(x) = f(-x)$$
for all x in the domain of f, and it is **odd** if
$$f(x) = -f(-x)$$
for all x in the domain of f.

- The graph of an even function is symmetric with respect to the y-axis, the graph of an odd function is symmetric with respect to the origin.

- It's easy to see that the derivative of an even function is odd and that of an odd function is even. Suppose f is even. Then we get
$$\begin{array}{rcll} f(x) & = & f(-x) & f \text{ is even} \\ f'(x) & = & -f'(-x) & f' \text{ is odd} \\ f''(x) & = & f''(-x) & f'' \text{ is even} \end{array}$$

- With the right choice of the integration constant we can also go the other way. (Think about what the word "right" means.)

- Thus we get the fact

The $\begin{array}{c}\textbf{derivative}\\\textbf{antiderivative}\end{array}$ of an $\begin{array}{c}\textbf{even}\\\textbf{odd}\end{array}$ function is $\begin{array}{c}\textbf{odd}\\\textbf{even}\end{array}$.

- A little **puzzle** for you: Compute

$$\int_{-\sqrt{\pi}}^{\sqrt{\pi}} \left(\sin x + \sin^{13} x^{17} \right)^{15} \mathrm{d}x =$$

- in general, if f is odd, then

$$\int_{-b}^{0} f(t)\mathrm{d}t = -\int_{0}^{b} f(t)\mathrm{d}t$$

and

$$\int_{-b}^{b} f(t)\mathrm{d}t = 0$$

- Examples:

$$\int_{-1}^{1} t^3\mathrm{d}t =$$

$$\int_{-1}^{1} \sin x^3\mathrm{d}x =$$

- The corresponding property for even functions is not nearly as useful:

$$\int_{-b}^{0} f(t)\mathrm{d}t = \int_{0}^{b} f(t)\mathrm{d}t$$

and

$$\int_{-b}^{b} f(t)\mathrm{d}t = 2\int_{0}^{b} f(t)\mathrm{d}t.$$

Integrals of periodic functions

- f is p-periodic if

$$f(t+p) = f(t)$$

  for all t in the domain of f.

- For example, sin and cos are 2π-periodic (and also, for example, 6π-periodic.
- The tan function is π-periodic.
- Useful property, not mentioned in the textbook: When you integrate a periodic function over an integer number of periods it does not matter where you start. Assuming f is p-periodic, we get

$$\int_a^{a+p} f(t)\mathrm{d}t = \int_{a+z}^{a+p+z} f(t)\mathrm{d}t$$

  for all real numbers z.

- Reminder: The integral may be clear geometrically, even if we are unable to find an antiderivative. One of the most frequently arising examples is

  $\int_{-r}^{r} \sqrt{r^2 - x^2} \mathrm{d}x =$

- Note: we don't yet know how to do this (and will learn more in Math 1220) but for your information and entertainment:

$$\int \sqrt{r^2 - x^2} \mathrm{d}x = \frac{1}{2} \left(x\sqrt{r^2 - x^2} + \arctan \frac{x}{\sqrt{r^2 - x^2}} \right) + C.$$

5.1 Computation of Area

- Setting the stage, more on Friday:
- We know that if $f(x) > 0$ for all x in $[a, b]$ then
$$\int_a^b f(x)\mathrm{d}x$$
is the area of the region enclosed by the x-axis, the graph of f, and the vertical lines $x = a$ and $x = b$.

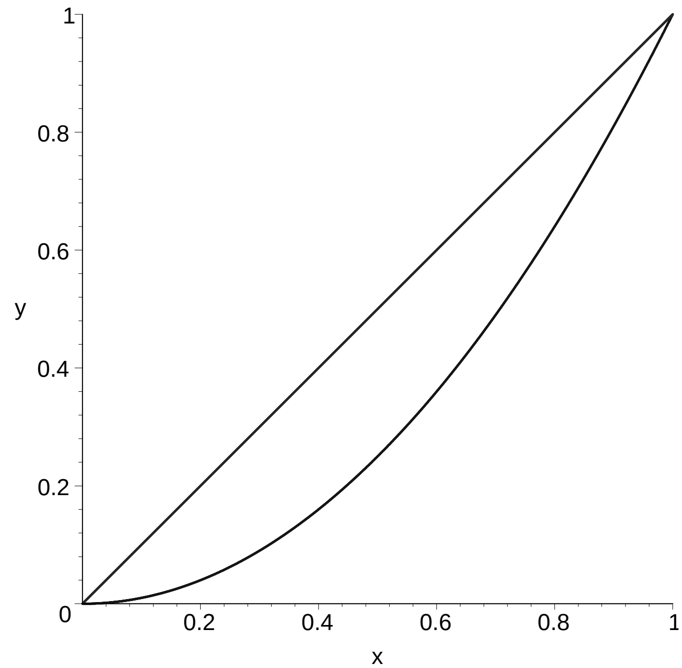

Figure 3. The region enclosed by yx and $y = x^2$.

- What about, for example, the region enclosed by the graphs of $y = x$ and $y = x^2$, as shown in Figure 3?

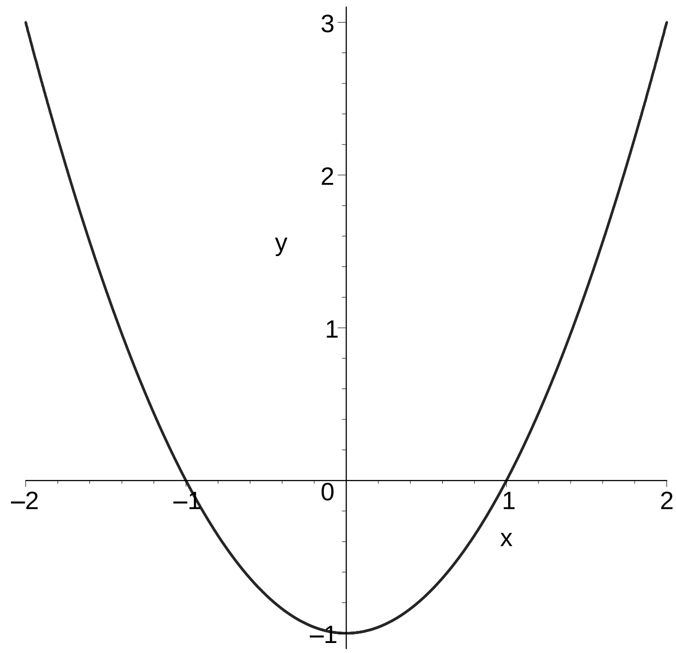

Figure 4. A three part region.

- Or, how about the region enclosed by the graphs of $y = x^2 - 1$, the x-axis, and the vertical lines $x = \pm 2$ shown in Figure 4?