$$\frac{d}{dx}x^3 = 3x^2 \qquad \int 3x^2 dx = x^3 + c$$

Notes of 3/26/24

Integration by Substitution

- Every differentiation rule comes with an integration rule, just go the other way.
- Integration by substitution is the inverse of the chain rule.

$$\int f\big(g(x)\big)g'(x)\mathrm{d}x = f\big(g(x)\big) + C$$

because, by the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}f\big(g(x)\big) = f\big(g(x)\big)g'(x).$$

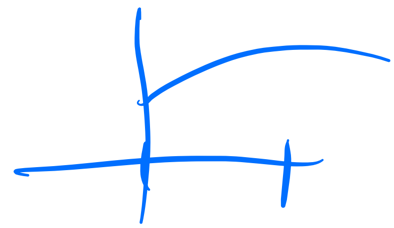
- You can carry out the integration either directly, by recognizing the pattern and the antiderivative, or, more elaborately, by using the **substitution**

$$u = g(x), \qquad \mathrm{d}u = \frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = g'(x)\mathrm{d}x.$$

- When computing indefinite integrals we need to return to the original variable.
- When computing definite integrals we can do that too, but we don't need to. If we stick with u as the variable we **need to change the limits of integration**.
- Some adjustments, like multiplying with suitable constants may be necessary.

- We'll do some examples now, and more next week during the review.
- $I = \int x \sin x^2 \mathrm{d}x = -\frac{1}{2}\cos x^2 + C$

CBD ✓



$u = x^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$x\,dx = \frac{1}{2}du$

$I = \frac{1}{2}\int \sin u\,du = -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos x^2 + C$

What about $I = \int_0^{\sqrt{\pi}} x \sin x^2 \mathrm{d}x =$

$I = \left[-\frac{1}{2}\cos x^2\right]_0^{\sqrt{\pi}} = +\frac{1}{2}\left[1+1\right] = 1$

$I = \frac{1}{2}\int_0^{\pi} \sin u\,du = -\frac{1}{2}\cos u\Big|_0^{\pi} = \frac{1}{2} + \frac{1}{2} = 1$

$\int_a^b f(x)dx = \left[F(x)\right]_a^b = F(b) - F(a)$

- Example 11, page 247
- $I = \int_{0 \, =x}^{\pi/4 \, =x} \sin^3 2x \cos 2x \mathrm{d}x = \frac{1}{2}\int_0^1 u^3 \, du = \frac{1}{2}\left[\frac{u^4}{4}\right]_0^1 = \frac{1}{8}$

$u = \sin 2x$

$du = (\cos 2x) \cdot 2 \, dx$

$\cos 2x \, dx = \frac{1}{2} du$

$\int u^3 = \frac{u^4}{4}$

$u = \sin 2x$

$x = \frac{\pi}{4} \quad u = \sin\frac{\pi}{2} = 1$

CBD

- Example 12, page 248
- $I = \int_0^1 \frac{x+1}{(x^2+2x+6)^2} \mathrm{d}x = \frac{1}{2}\int_6^9 \frac{1}{u^2}\,du = -\frac{1}{2}\frac{1}{u}\Big|_6^9$

$$= -\frac{1}{2}\left(\frac{1}{9}-\frac{1}{6}\right)$$

$$= -\frac{1}{2}\frac{6-9}{54} = \frac{1}{36}$$

$$u = x^2+2x+6$$

$$du = 2(x+1)\,dx$$

$$(x+1)\,dx = \frac{1}{2}du$$

$x=0 \quad u=6$

$x=1 \quad u=9$

$$\int \frac{1}{u^2}du = \int u^{-2}du$$

$$= -u^{-1} + C$$

$$\frac{d}{dx}x^p = px^{p-1}$$

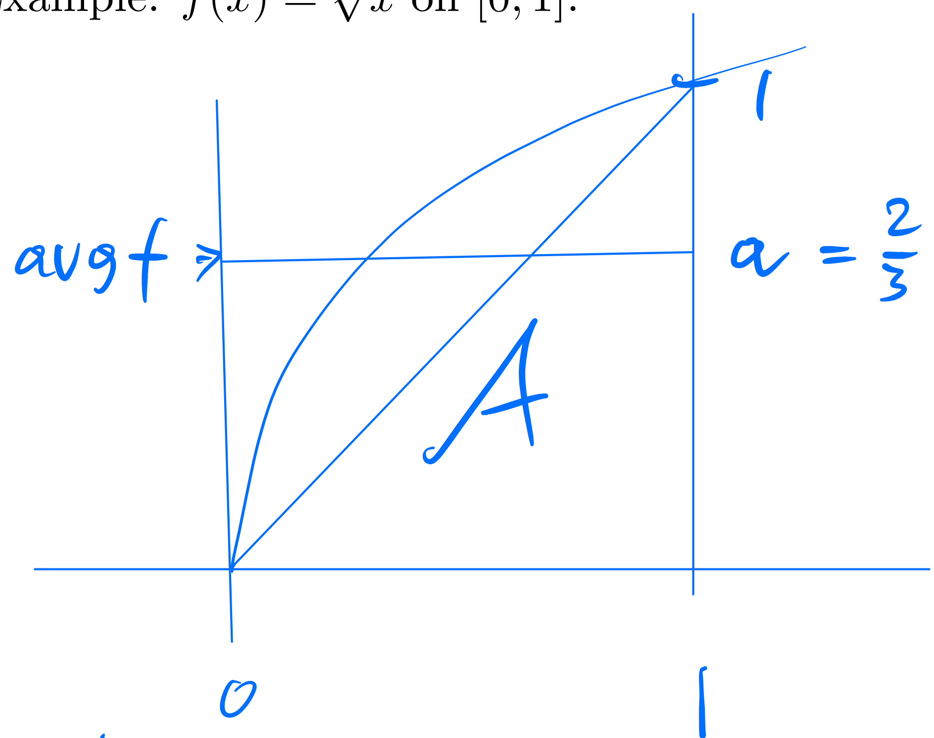
$$\int x^p dx = \frac{x^{p+1}}{p+1} + C$$

$$-\frac{1}{2}\frac{1}{u} = -\frac{1}{2}\frac{1}{x^2+2x+6}\Big|_0^1$$

$$= -\frac{1}{2}\left(\frac{1}{9}-\frac{1}{6}\right)$$

Average Value of a Function on an Interval

- Example: $f(x) = \sqrt{x}$ on $[0,1]$.



$$A = \int_0^1 x^{1/2}\,dx = a(1-0)$$

$$= \frac{2}{3} x^{3/2} \Big|_0^1 = \frac{2}{3}$$

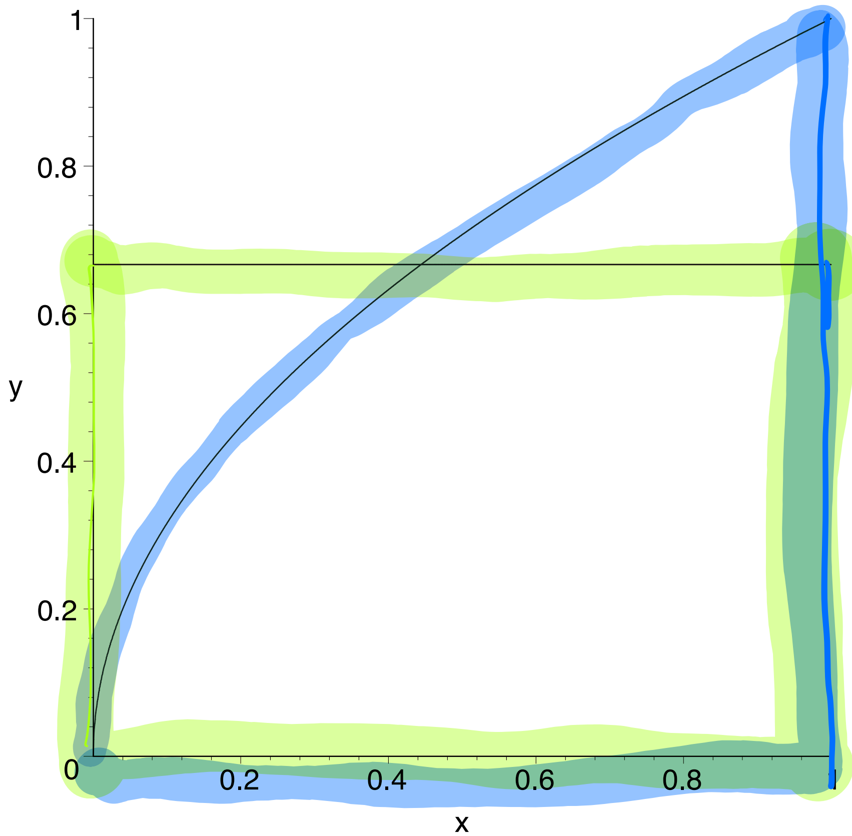
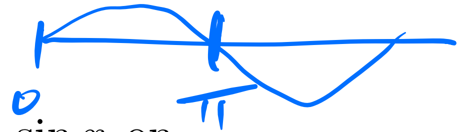


Figure 1. $f(x) = \sqrt{x}$ on $[0, 1]$.

$$\mathbf{avg}\sqrt{x} = \frac{\int_0^1 \sqrt{x}\mathrm{d}x}{1-0} = \frac{2}{3}$$



- Example: Average Value of $f(x) = \sin x$ on $[0, \pi]$

$$\int_0^{\pi} \sin x \, dx = -\cos x \Big|_0^{\pi} = 1 + 1 = 2$$

$$\text{avg } f = \frac{2}{\pi}$$

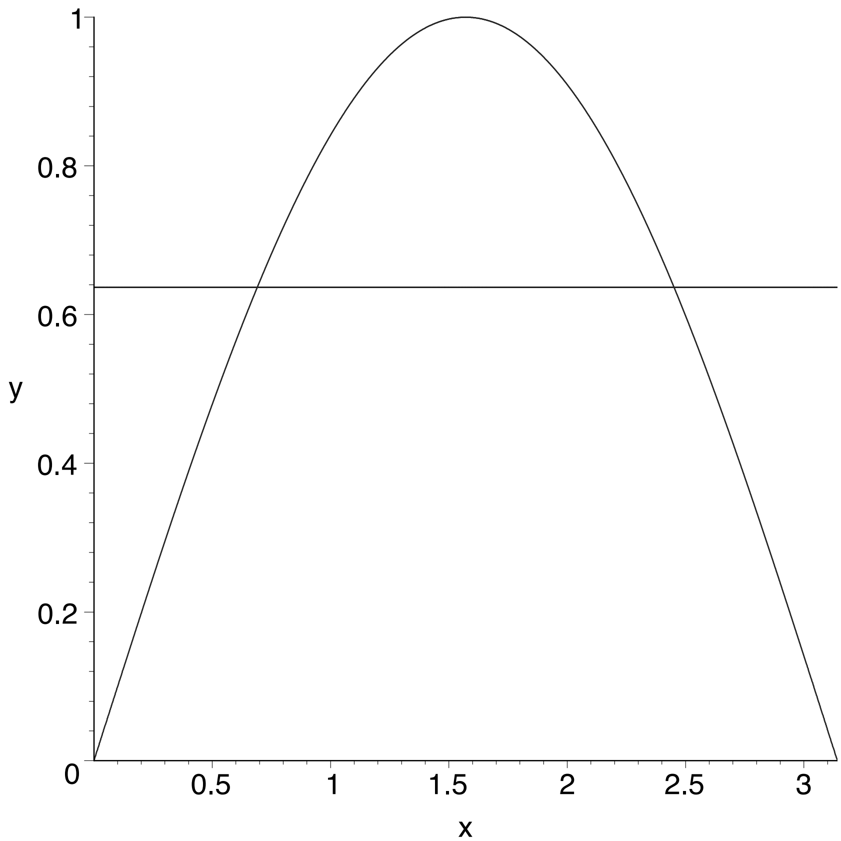


Figure 2. Average Value of $f(x) = \sin x$ on $[0, \pi]$.

$$\mathbf{avg} \sin x = \frac{\int_0^\pi \sin x \mathrm{d}x}{\pi - 0} = \frac{2}{\pi}$$

- In general the **average of f on** $[a, b]$ is defined as

$$\mathbf{avg} f = \frac{\int_a^b f(x)\mathrm{d}x}{b-a}$$

- Think of it as a thin sheet of ice of the shape defined by f melting and forming a rectangle.

The Mean Value Theorem for Integrals

- Recall the

  Mean Value Theorem for Derivatives:

  If f is differentiable on (a,b) and continuous on $[a,b]$ then there exists a c in (a,b) such that

  $$f'(c) = \frac{f(b)-f(a)}{b-a}$$

  or

  $$f'(c)(b-a) = f(b) - f(a).$$

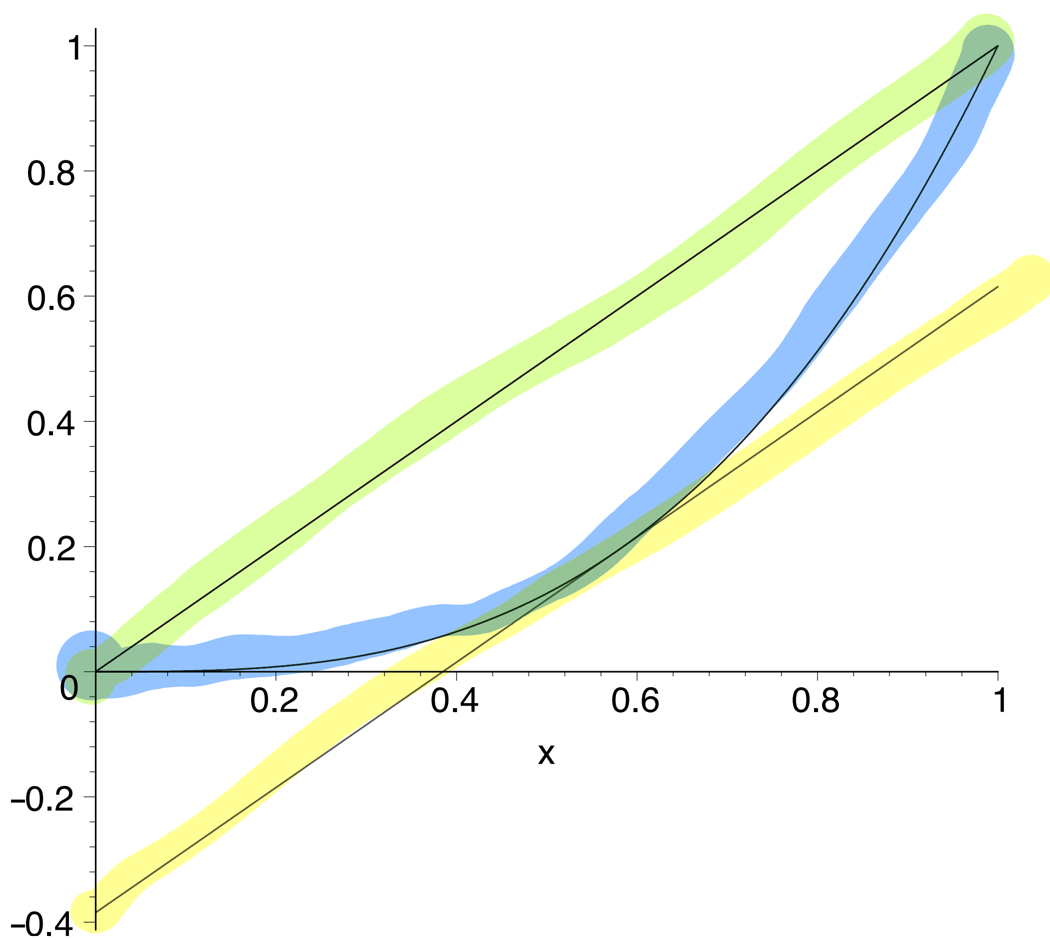


Figure 3. The Mean Value Theorem for Derivatives.

- The MVT for Integrals says that if f is continuous on $[a, b]$ there must be a point c in (a, b) such that f at that point equals the average value.
- That seems geometrically obvious.
- Stated more formally we have:
- Suppose f is continuous on $[a, b]$. Then there exists a number c in (a, b) such that

$$f(c) = \frac{1}{b-a} \int_a^b f(t)\mathrm{d}t.$$

- This can be rewritten as

$$F'(c)(b-a) = f(c)(b-a) = \int_a^b f(t)\mathrm{d}t = F(b)-F(a)$$

- Note that in particular,

$$f(c)(b-a) = F(b) - F(a)$$

is just the mean value value theorem for derivatives applied to the function F.

- Example: Compute c for $f(x) = x^p$ on the interval $[0, 1]$.

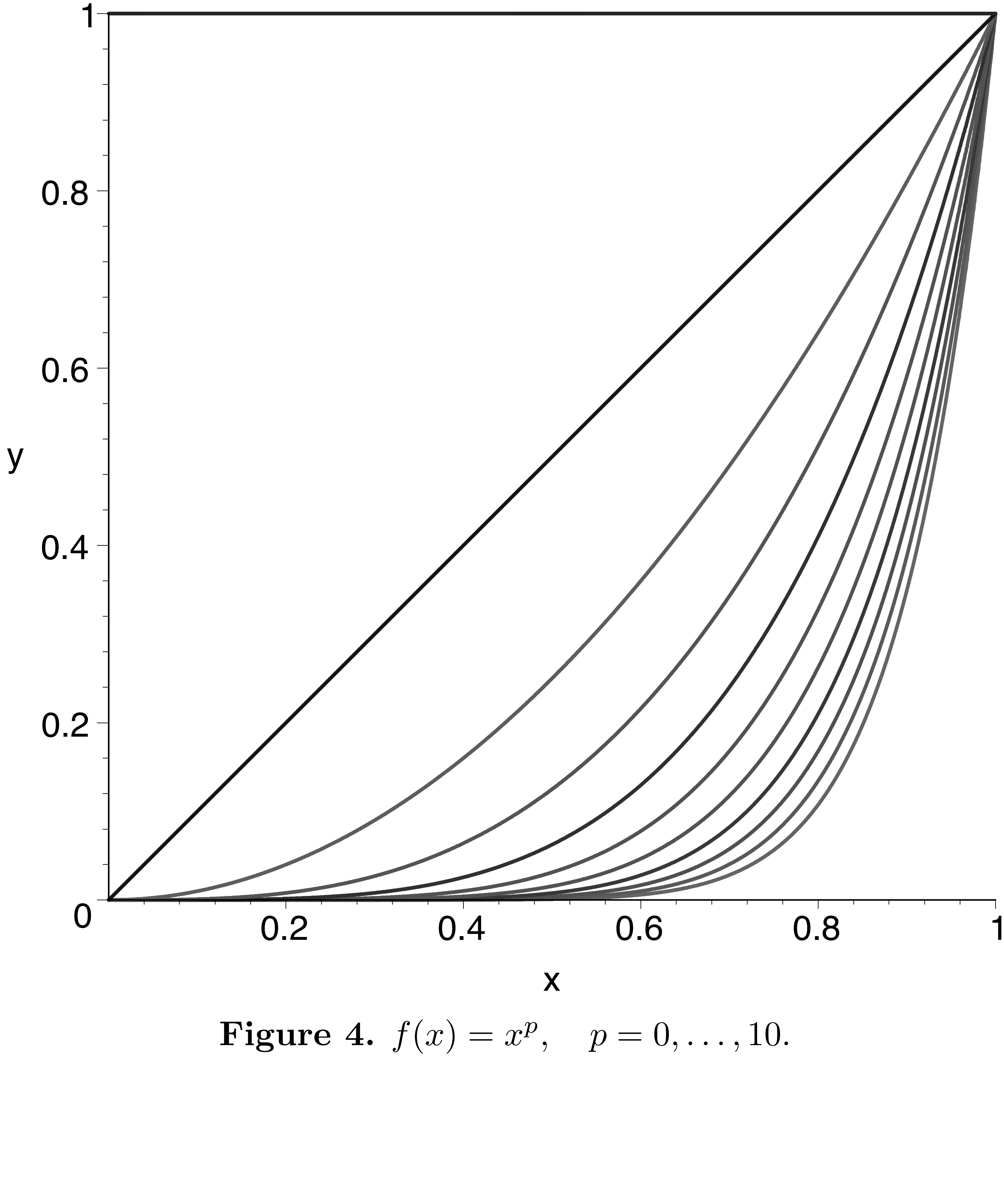


Figure 4. $f(x) = x^p, \quad p = 0, \ldots, 10.$

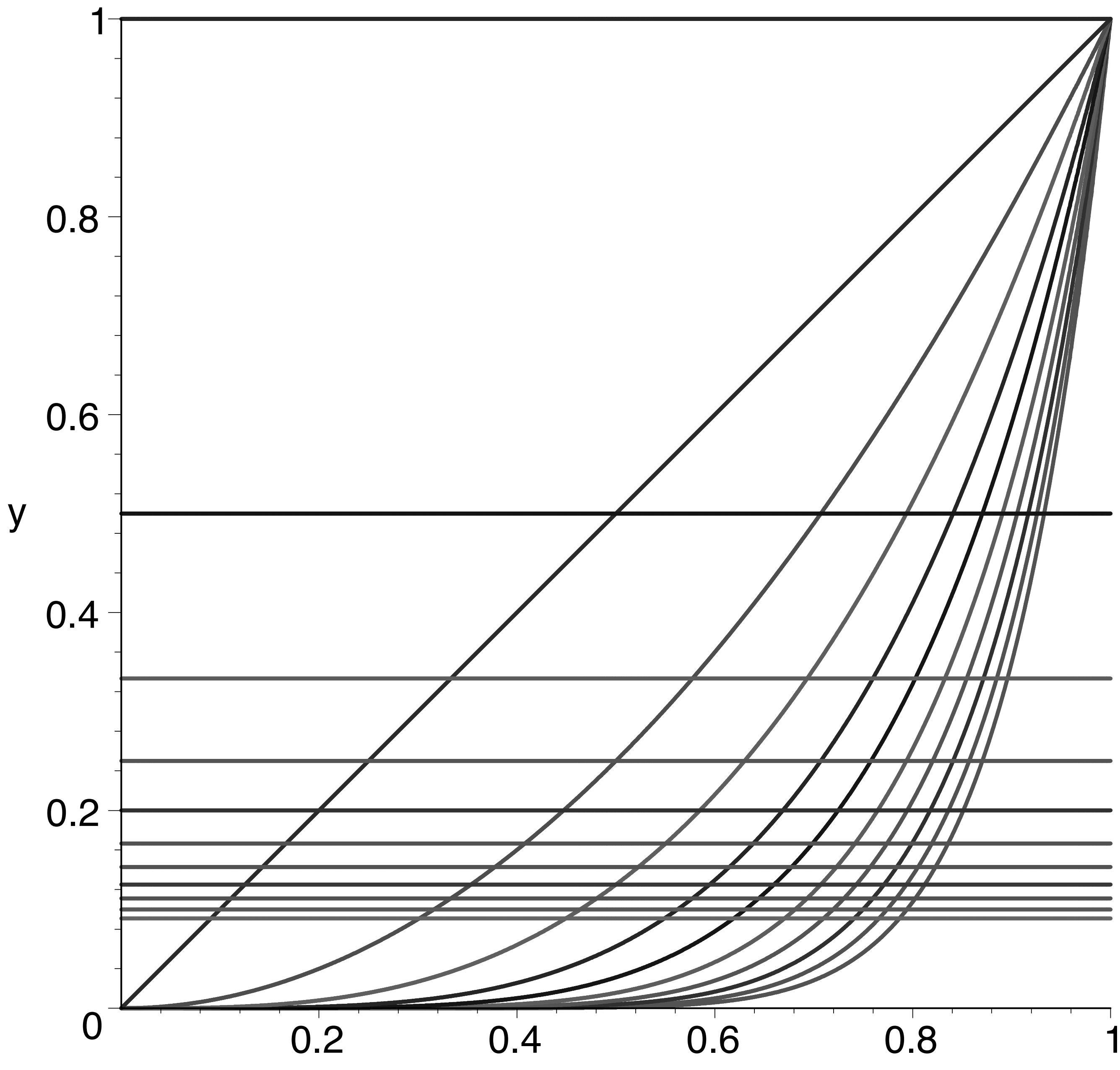


Figure 5. $f(x) = x^p, \quad p = 0, \ldots, 10$ and average values.