Notes of 3/26/24

Integration by Substitution

- Every differentiation rule comes with an integration rule, just go the other way.
- Integration by substitution is the inverse of the chain rule.

$$\int f\big(g(x)\big)g'(x)\mathrm{d}x = f\big(g(x)\big) + C$$

because, by the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}x} f\big(g(x)\big) = f\big(g(x)\big)g'(x).$$

- You can carry out the integration either directly, by recognizing the pattern and the antiderivative, or, more elaborately, by using the **substitution**

$$u = g(x), \qquad \mathrm{d}u = \frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = g'(x)\mathrm{d}x.$$

- When computing indefinite integrals we need to return to the original variable.
- When computing definite integrals we can do that too, but we don't need to. If we stick with u as the variable we **need to change the limits of integration**.
- Some adjustments, like multiplying with suitable constants may be necessary.

- We'll do some examples now, and more next week during the review.
- $I = \int x \sin x^2 \mathrm{d}x =$

What about $I = \int_0^{\sqrt{\pi}} x \sin x^2 \mathrm{d}x =$

- Example 11, page 247
- $I = \int_0^{\pi/4} \sin^3 2x \cos 2x \mathrm{d}x =$

- Example 12, page 248
- $I = \int_0^1 \frac{x+1}{(x^2+2x+6)^2} \mathrm{d}x =$

Average Value of a Function on an Interval

- Example: $f(x) = \sqrt{x}$ on $[0, 1]$.

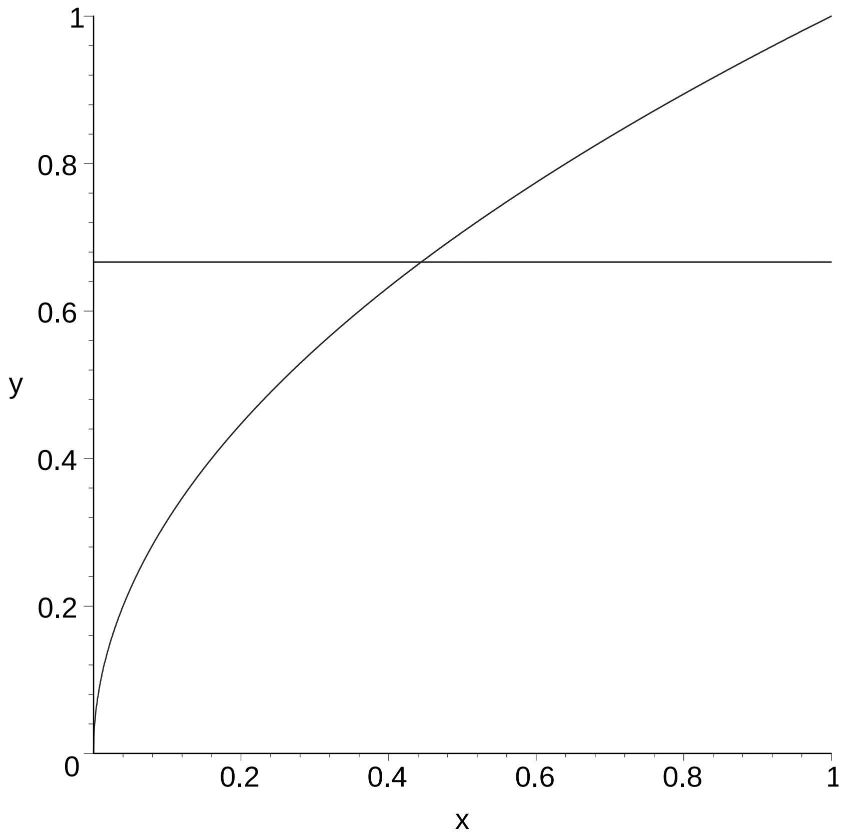

Figure 1. $f(x) = \sqrt{x}$ on $[0, 1]$.

$$\mathbf{avg}\sqrt{x} = \frac{\int_0^1 \sqrt{x}\mathrm{d}x}{1-0} = \frac{2}{3}$$

- Example: Average Value of $f(x) = \sin x$ on $[0, \pi]$

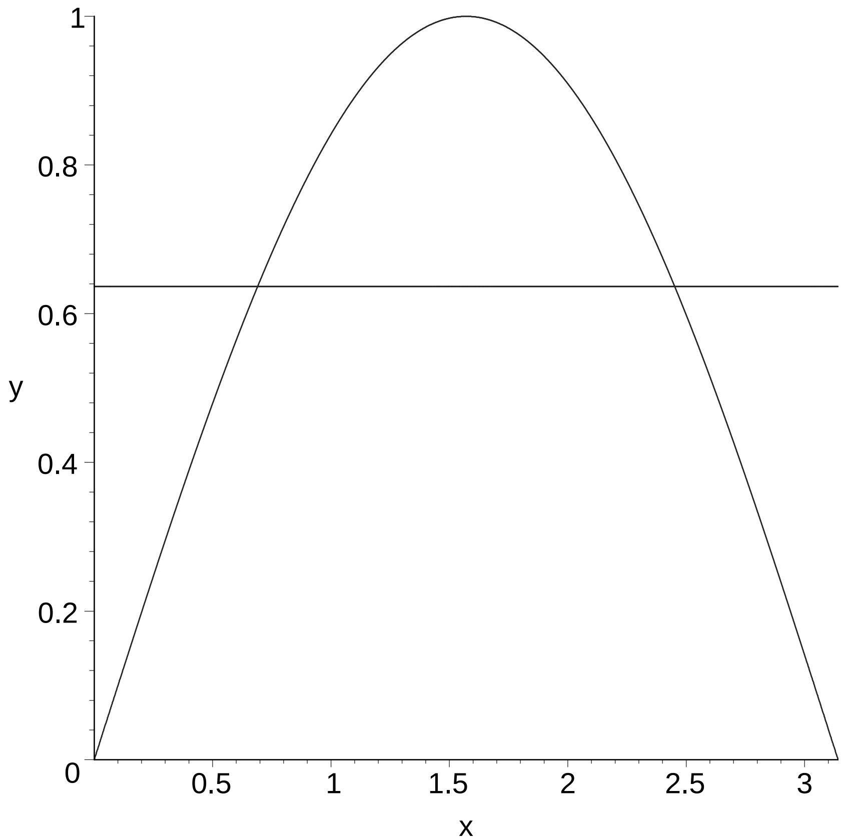

Figure 2. Average Value of $f(x) = \sin x$ on $[0, \pi]$.

$$\mathbf{avg} \sin x = \frac{\int_0^\pi \sin x \mathrm{d}x}{\pi - 0} = \frac{2}{\pi}$$

- In general the **average of f on** $[a,b]$ is defined as

$$\mathbf{avg}f = \frac{\int_a^b f(x)\mathrm{d}x}{b-a}$$

- Think of it as a thin sheet of ice of the shape defined by f melting and forming a rectangle.

The Mean Value Theorem for Integrals

- Recall the

  Mean Value Theorem for Derivatives:

  If f is differentiable on (a, b) and continuous on $[a, b]$ then there exists a c in (a, b) such that
  $$f'(c) = \frac{f(b) - f(a)}{b - a}$$
  or
  $$f'(c)(b - a) = f(b) - f(a).$$

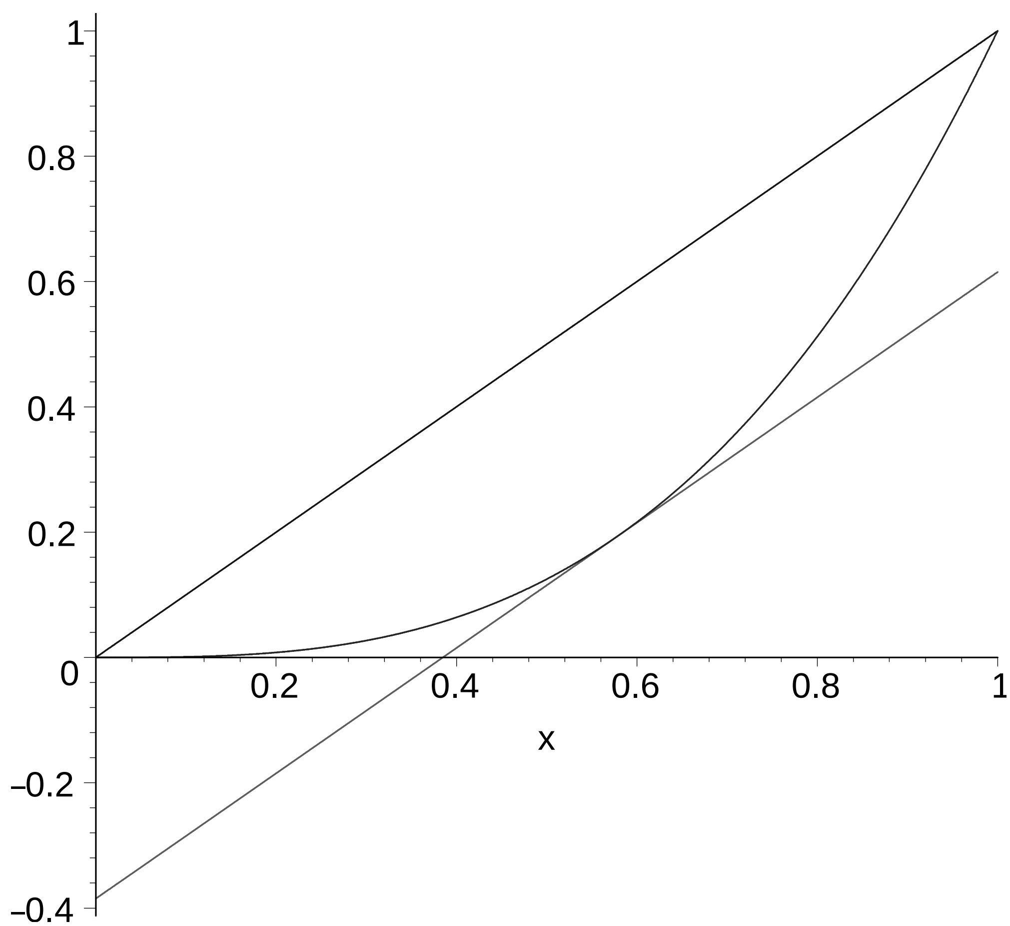

Figure 3. The Mean Value Theorem for Derivatives.

- The MVT for Integrals says that if f is continuous on $[a, b]$ there must be a point c in (a, b) such that f at that point equals the average value.
- That seems geometrically obvious.
- Stated more formally we have:
- Suppose f is continuous on $[a, b]$. Then there exists a number c in (a, b) such that

$$f(c) = \frac{1}{b-a} \int_a^b f(t) \mathrm{d}t.$$

- This can be rewritten as

$$F'(c)(b-a) = f(c)(b-a) = \int_a^b f(t) \mathrm{d}t = F(b) - F(a)$$

- Note that in particular,

$$f(c)(b-a) = F(b) - F(a)$$

is just the mean value value theorem for derivatives applied to the function F.

- Example: Compute c for $f(x) = x^p$ on the interval $[0, 1]$.

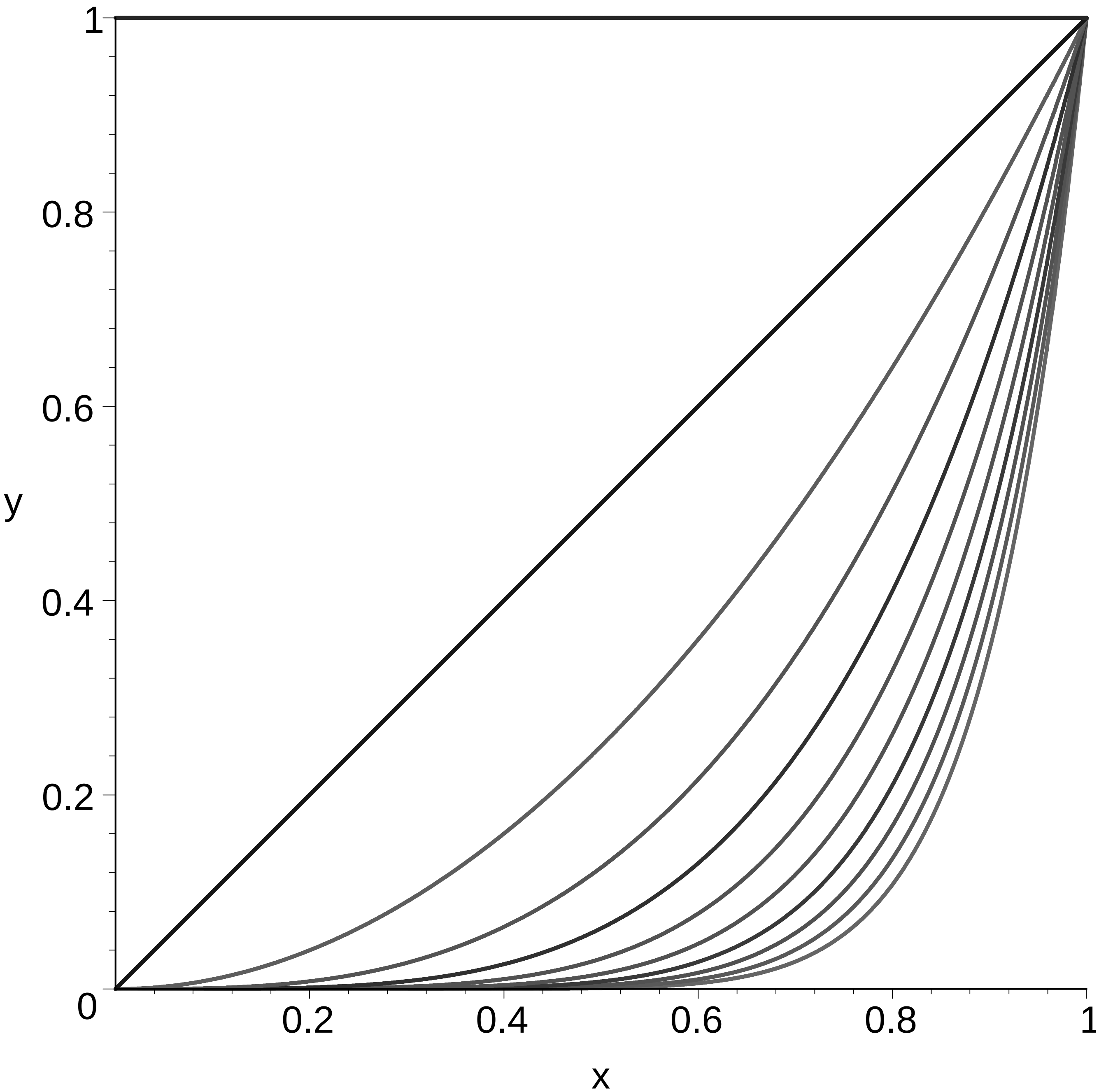

Figure 4. $f(x) = x^p, \quad p = 0, \ldots, 10.$

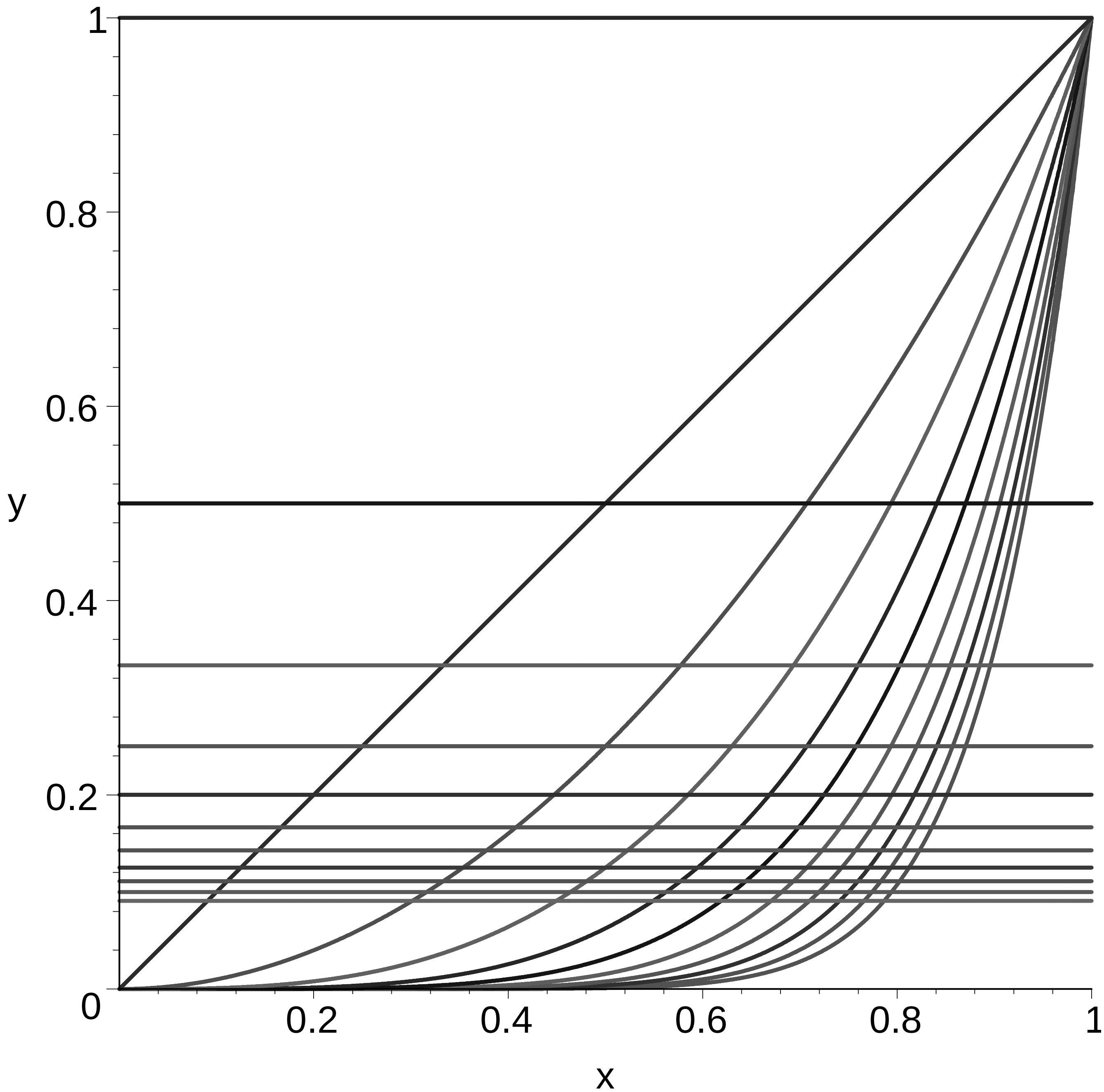

Figure 5. $f(x) = x^p, \quad p = 0, \ldots, 10$ and average values.