Announcements

• You should have received an email from me with ww login info.

• If not check your spam filter.

• If you still can’t find the info send a message to me at pa@math.utah.edu

• It is crucial that ww has an email address for you that you monitor regularly. You can log into ww and change your email address to anything you like.

• When you have a query about a specific ww question please use the “Email Instructor” link on the bottom of the problem page. That way I will get much more information than if you send me a regular email. In particular I get a link that lets me see the problem page exactly as you see it.

• When using the ww link, leave your best guess of the answer in the answer field until I see it and respond to you. I may get a clue about what’s going wrong that way.

• Of course you can send me regular email on any other subject.

• If you want to know what you did wrong in a calculation of course I need to see that calculation. You can type it, or send me a scan or a photograph.

• The second home work is now open.

• You should be done with the first hw (many people are), but it will only close on Tuesday, one minute before midnight.

• Do check your email regularly.
Review

• We will begin by quickly reviewing some of the prerequisites for this class.

• These are subjects that you are already familiar with.

• We will skip many explanations and go much faster than we will go when we enter new territory (i.e., Calculus proper).

The Real Number System

• the natural numbers are

\[1, 2, 3, 4, \ldots\]

• Computer Scientists, and some mathematicians, like to start with zero.

• We have four arithmetic operations, addition, subtraction, multiplication, and division.

• Certain rules hold, such as the commutative and associative laws of addition and multiplication, and the distributive law

\[a(b + c) = ab + ac\]

• The sum and product of two natural numbers is a natural number. We say that the natural numbers are closed under addition and multiplication.

• The basic idea of building the number system is to enlarge it so that the system is closed under more operations, and so that the basic rules continue to hold!

• To make subtraction always possible we extend the system to the integers
\[ \ldots -3, -2, -1, 0, 1, 2, 3, \ldots \]

- The integers are closed under addition, multiplication, and subtraction.

- Note that all natural numbers are integers, but not all integers are natural numbers.

- To make division always possible we extend the number system again, to the **rational numbers** (or fractions). A rational number is a quotient of the form

\[ \frac{p}{q} \]

where \( p \) and \( q \) are integers, and \( q \neq 0 \).

- It is crucial that you understand how to add, subtract, multiply, and divide rational numbers, because the exact same rules hold for more general mathematical expressions! If you can’t do fractions you can’t do algebra!

- We can’t divide by zero. Why not?

\[
    x = \frac{p}{q} \\
    qx = p \\
    q = 0
\]

\[
    0 = 0x = p \\
    p \neq 0 \\
    p = 0 \\
    \text{any } x
\]
Real Numbers

- The Greek discovered more than 2000 years ago that some numbers aren’t rational. For example, $\sqrt{2}$, $\pi$, and $e$ aren’t rational.

- Note that the sets of natural numbers, integers, rational numbers, real numbers, are nested.

Figure 1. The Real Number System.
A Little Bit of Logic

Consider the true statement $S$

**All integers are rational numbers**

- We can also write this as

  
  
  $x$ is integer $\implies x$ is rational.

- The symbol $\implies$ is pronounced **implies**.

- In general, if $P$ and $Q$ are statements we write that

  
  
  $P \implies Q$

  
  and say that $P$ implies $Q$, or $Q$ is implied by $P$.

- We also say that $P$ is a **sufficient** condition for $Q$ and $Q$ is a **necessary condition** for $P$.

- The **converse** of $P \implies Q$ is $Q \implies P$

- The converse of $S$ (**All integers are rational numbers.**) is **All rational numbers are integer**. That statement is of course false.

- The **negation** of a statement is the assertion that the statement is false. Here are two (of many more) ways of negating the statement $S$.

  - It is not true that all integers are rational numbers.
  - Some integers are not rational numbers.

- Both of these statements are false, of course. Indeed, the negation of a true statement is always false, and the negation of a false statement is always true.

- The negation of a statement $P$ is sometimes written as $\sim P$ (pronounced **not** $P$).
• A very useful, but subtle concept, is that of the **contrapositive** of a statement. The contrapositive of

\[ P \implies Q \]

is

\[ \sim Q \implies \sim P. \]

• For example, the contrapositive of the statement \( S \) (All integers are rational numbers) is

\[ \text{If } x \text{ is not rational, it's not integer} \]

• Two statements are **equivalent** if each implies the other. To state that \( P \) and \( Q \) are equivalent we write

\[ P \iff Q \]

• For example, let \( P \) be the statement that \( x \) is even and \( Q \) the statement that \( x \) is integer and divisible by 2. Then \( P \) and \( Q \) are equivalent.

• Two equivalent statements are either both true or both false.

• A statement and its contrapositive are equivalent:

\[ P \implies Q \iff \sim Q \implies \sim P. \]
• Let’s try these concepts on the (false) statement

   All dogs are brown

• Converse:

   if it's brown it's a dog

• Negation:

   Not all dogs are brown

• Contrapositive:

   If it's not brown it's no dog
Irrationality of \( \sqrt{2} \)

- Here is an example of a logical argument showing that the square root of 2 is irrational. I think it’s one of the prettiest thoughts in human history!

\[
\sqrt{2} = \frac{p}{q} \quad q \neq 0
\]

\( p, q \) not both even

\[
\sqrt{2} = \frac{p}{q} \quad \Rightarrow \quad \left( \frac{p}{q} \right)^2 = 2
\]

\[
\Rightarrow \quad \frac{p^2}{q^2} = 2 \quad \land \quad q^2
\]

\[
\Rightarrow \quad p^2 = 2q^2
\]

\( p^2 \) is even

\( p \) is even

\( p^2 \) is divisible by 4

\( 2q^2 \) is divisible by 4

\( q^2 \) is even

\( q \) is even

\[
p^2 = p \cdot p = (2r)(2r) = 4r^2
\]
Inequalities, Intervals, Absolute Values

• Suppose $a, b, x$ are real numbers.

• the following notations should be familiar.

\[ a < b, \quad a \leq b, \quad a > b, \quad a \geq b, \quad a = b, \quad a \neq b \]

• Similarly, you should be familiar with these interval notations:

\[
[a, b] \quad (a, b) \quad (a, b] \quad [a, b)
\]
\[
(\infty, b) \quad (\infty, b] \quad (a, \infty) \quad [a, \infty)
\]
\[
(\infty, \infty)
\]

• Another notation for \((\infty, \infty)\) (the set of all real numbers) is \(\mathbb{R}\).

• The absolute value function is defined by

\[
|x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0 
\end{cases}
\]
Expressions and Equations

- An **expression** is a collection of variables and constants connected by mathematical operations.

- **Evaluating an expression** means substituting constants or other expressions for variables.

- Two expressions are **equivalent** if evaluating them (at the same values of the variables) always gives identical answers.

- **Simplifying an expression** means replacing it by a suitable equivalent expression. The meaning of the word **simple** depends on the context (and sometimes on personal taste).

- An **equation** consists of two expressions separated by an equals sign. For example, $3x + 4 = 13$ is an equation. It is true only when $x = 3$.

- An **identity** is an equation that is true for all values of its variables. For example

$$a + b = b + a$$

and

$$\cos^2 x + \sin^2 x = 1$$

are identities.

**Solving Equations**

- To solve an equation means to figure out for which values of its variables it is true. We say that those values **satisfy the equation**.

- Solving equations is a big subject.

- However, there is just one principle: Apply the same operation on both sides of the equation until you have the variable by itself.

- How to pick the operations is the crux of the matter, of course, and depends on the context.
Examples

- How not to solve

\[ \frac{x^2}{x+2} = (x-1)(x+2) \]

- How to solve

\[ \frac{x^2}{x+2} = x - 1 \]

\[ \frac{x^2(x+2)}{(x+2)} = (x-1)(x+2) \]  
\[ x^2 = (x-1)(x+2) \]  
\[ x^2 = x^2 + 2x - x - 2 \]  
\[ x^2 = x^2 + x - 2 \]  
\[ 0 = x - 2 \]  
\[ x = 2 \]

check: \[ \frac{2^2}{2+2} = \frac{4}{4} = 1 = 2 - 1 \]