Math 1210-23

Notes of 3/25/24

Announcement

- Next week, Friday: Exam 3 on chapters 3 and 4. Last hw covering those topics is hw 11 which is now open. This will be the last midterm exam. The final will be comprehensive and cover the whole semester about evenly.

4.3-4 More on the FToC

- FToC = Fundamental Theorem of Calculus
- Recall the two versions of the **Fundamental Theorem of Calculus**:

$$\frac{\mathrm{d}}{\mathrm{d}x} \underbrace{\int_a^x f(t)\mathrm{d}t}_{A(x)} = f(x) \quad \text{and}$$

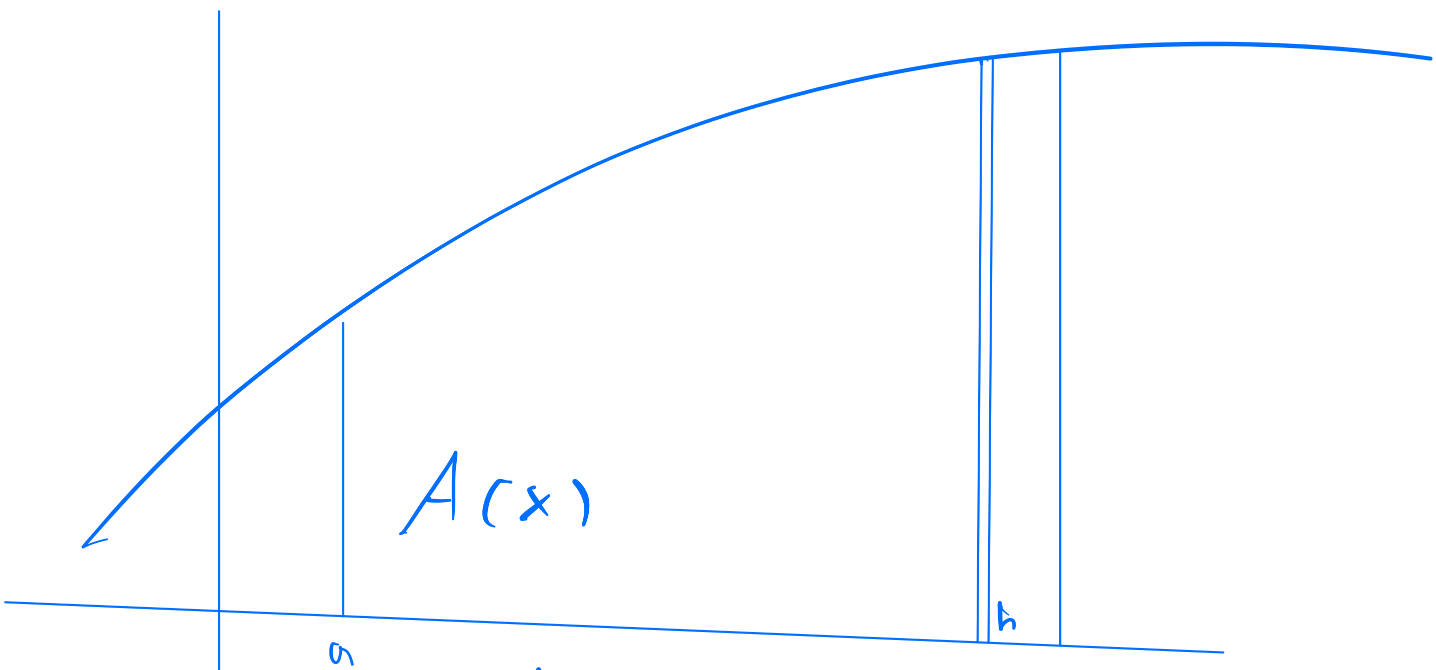
$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$$

where, throughout today, and most of the time in general, F is an antiderivative of f:

$$F' = f$$

- The FToC is one of the most important and profound parts of Calculus. The issue was thought about by the ancient Greeks, but it took the human species some 2000 years to figure out the FToC. (So it's OK if it takes you a couple of days :)

- To make sure we all understand the FToC and are on the same page, let me go again over why it is true, and why the two versions are equivalent.



$A(a) = 0$

$A(x) = \int_a^x f(t)dt = F(x) - F(a)$

- We have the following notations:

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a) = [F(x)]_a^b = [F(x)]_{x=a}^{x=b} = F(x)|_a^b = F(x)|_{x=a}^{x=b}$$

Examples

- $\int_0^1 x^p \mathrm{d}x =$ $\dfrac{x^{p+1}}{p+1}\Big|_0^1 = \dfrac{1}{p+1}$

  $c\int_a^b f(t)dt = \int_a^b c\, f(t)dt$

  CBD

- $\frac{\mathrm{d}}{\mathrm{d}s}\int_a^s \sin t^2 \mathrm{d}t =$ $\sin s^2$

- $\frac{\mathrm{d}}{\mathrm{d}s}\int_a^{\sin s} \sin t^2 \mathrm{d}t \neq$ $\sin(\sin^2 s)$

  $F'(t) = \sin t^2$

  $= \dfrac{d}{ds}\Big(F(\sin s) - F(a)\Big) = F'(\sin s)\cos s$

  $= \sin(\sin^2 s)\cos s$

$\int_0^2 x \mathrm{d}t =$ $x\int_0^2 1\, dt = x[t]_0^2 = 2x$

Suppose T is constant. Compute $\frac{\mathrm{d}}{\mathrm{d}x}\int_0^T t^t \mathrm{d}t =$ 0

Suppose T is a differentiable function of x. Compute $\frac{\mathrm{d}}{\mathrm{d}x}\int_0^T t^t \mathrm{d}t =$ $T^T \cdot \dfrac{dT}{dt}$

↳ WTI

More General Differentiation

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{a(x)}^{b(x)} f(t)\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}x}\Big(F\big(b(x)\big) - F\big(a(x)\big)\Big)$$

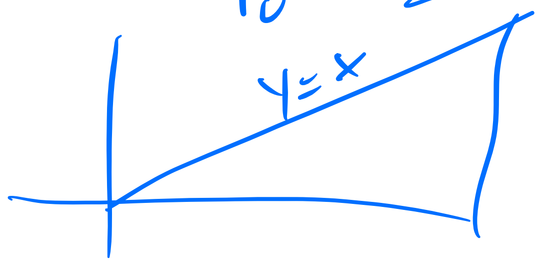
$$= F'\big(b(x)\big)b'(x) - F'\big(a(x)\big)a'(x)$$

$$= f\big(b(x)\big)b'(x) - f\big(a(x)\big)a'(x)$$

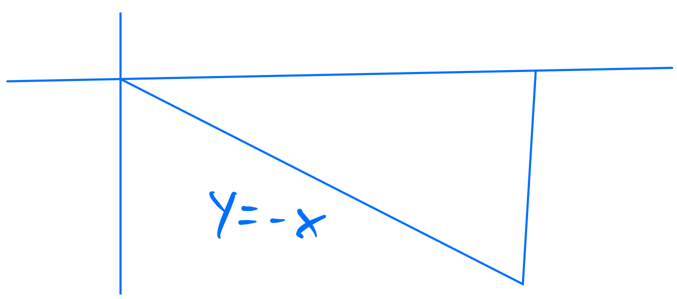
- Example

  $\frac{\mathrm{d}}{\mathrm{d}x}\int_{\sin x}^{\cos x} \sin t \mathrm{d}t =$ $- \sin(\cos x) \sin x - \sin(\sin x) \cos x$

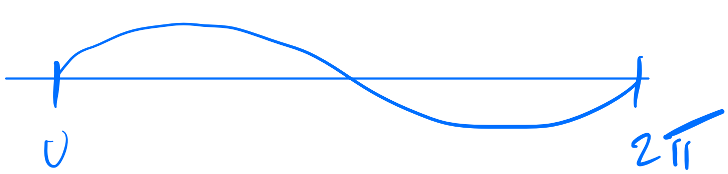
- We started by thinking of definite integrals as areas. That interpretation requires the integrand to be non-negative.
- $\int_0^1 x\mathrm{d}x = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$



- But consider instead:
- $\int_0^1 -x\mathrm{d}x = -\frac{x^2}{2}\Big|_0^1 = -\frac{1}{2}$



- We get the negative of the area. Regions underneath the x-axis have a "negative area".
- $\int_0^{2\pi} \sin x\mathrm{d}x = 0 = -\cos x\Big|_0^{2\pi} = -1 - (-1) = 0$



- The areas above and below the x axis cancel.

- Note that we don't have to interpret the integral as area. For example, if $f(t)$ gives the velocity at time t then the integral is distance. This is because for constant velocity distance equals time times velocity.

Switching Limits of Integration

- $\int_1^2 x^2 + x\mathrm{d}x =$ $\frac{x^3}{3} + \frac{x^2}{2}\Big|_1^2 = \frac{8}{3} + 2 - \left(\frac{1}{3} + \frac{1}{2}\right)$
  $= \frac{7}{3} + \frac{3}{2} = \frac{14+9}{6} = \frac{23}{6}$

- $\int_2^1 x^2 + x\mathrm{d}x =$ $\frac{x^3}{3} + \frac{x^2}{2}\Big|_2^1 = \left(\frac{1}{3} + \frac{1}{2}\right) - \left(\frac{8}{3} + 2\right) = -\frac{23}{6}$

- in general

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$$

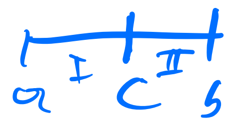
$$= -\big(F(a) - F(b)\big)$$

$$= -\int_b^a f(x)\mathrm{d}x$$

$$\boxed{\int_a^b f(x)\mathrm{d}x = -\int_b^a f(x)\mathrm{d}x}$$

- We discussed

$$I = \boxed{\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x}$$



- This makes geometric sense if

$$a < c < b. \qquad (1)$$

- It also follows from the FToC:

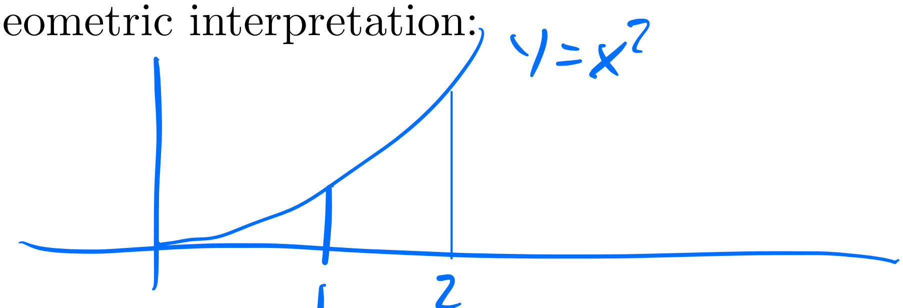
$$F(b) - F(a) = \cancel{F(c)} - F(a) + F(b) - \cancel{F(c)}$$

- However, we don't need the assumption (1).
- For example:

$$\int_0^1 x^2\mathrm{d}x = \int_0^2 x^2\mathrm{d}x + \int_2^1 x^2\mathrm{d}x.$$

$$= \frac{x^3}{3}\Big|_0^2 + \frac{x^3}{3}\Big|_2^1 = \frac{8}{3} + \frac{1}{3} - \frac{8}{3} = \frac{1}{3}$$

- Geometric interpretation:



Integrals May Not Exist

Example

$$\int_0^1 \frac{1}{x^2}\mathrm{d}x \quad = -\frac{1}{x}\Big|_0^1 = -\frac{1}{1} + \frac{1}{0}$$

does not exist, even though we can evaluate the antiderivative at the limits of integration and take the difference.

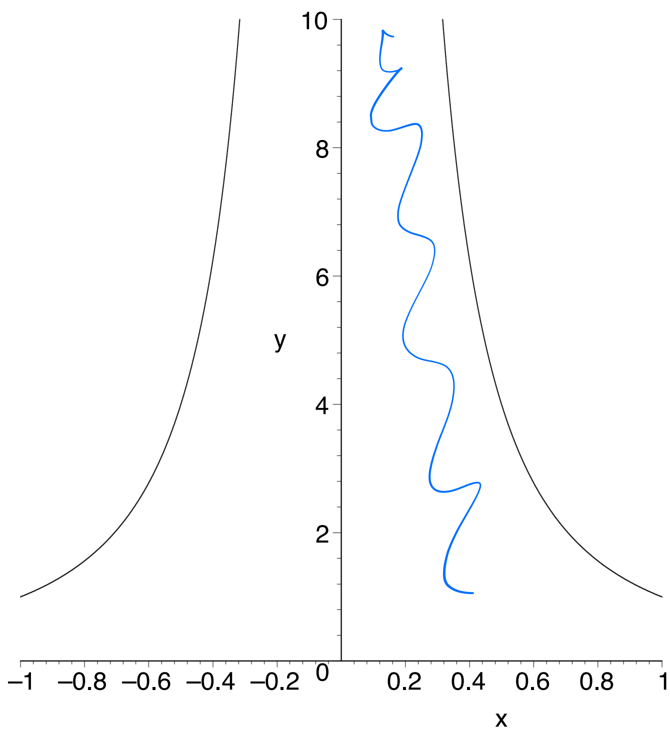
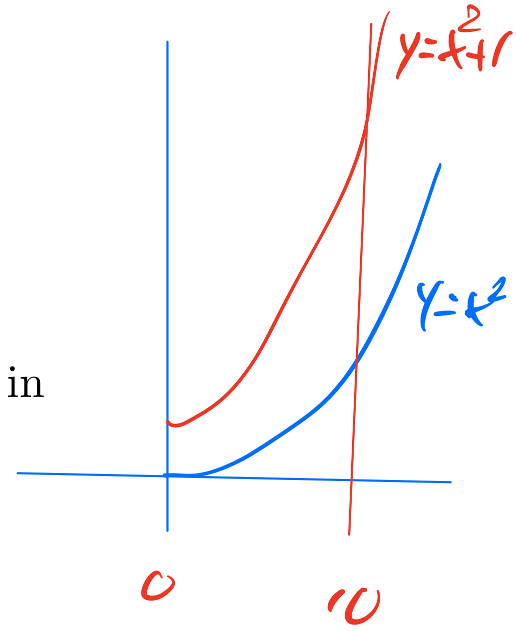


Figure 1. Graph of $y = 1/x^2$.

- However, all continuous functions are integrable.
- Many discontinuous functions are too, but we leave this to another day.

Comparison Property



- See page 235, textbook.
- Which is larger, $\int_0^{10} x^2 \mathrm{d}x$, or $\int_0^{10} x^2 + 1 \mathrm{d}x$
- In general: Suppose $f(x) \le g(x)$ for all x in $[a, b]$ (where $a < b$). Then

$$\int_a^b f(x)\mathrm{d}x \le \int_a^b g(x)\mathrm{d}x.$$

  (See Theorem B, page 235).
- This follows, for example, from the Riemann Sum definition of the integral.
- Consequence: Suppose

$$m \le f(x) \le M$$

  for all x in $[a, b]$. Then

$$\int_a^b m\mathrm{d}x \le \int_a^b f(x)\mathrm{d}x \le \int_a^b M\mathrm{d}x,$$

  i.e.,

$$m(b-a) \le \int_a^b f(x)\mathrm{d}x \le M(b-a).$$

  (See Theorem C, page 236).

Linearity

- Major Property (See Theorem D, page 236).

$$\int_a^b kf(x)\mathrm{d}x = k\int_a^b f(x)\mathrm{d}x$$

and

$$\int_a^b f(x) + g(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}x.$$

Integration by Substitution

- Inverse Process of the Chain Rule
- $\int x \sin x^2 \mathrm{d}x = \frac{1}{2} \int \sin u \, du = -\frac{1}{2} \cos u + C'$

$u = x^2$

$\frac{du}{dx} = 2x$

$du = 2x \, dx$

$x \, dx = \frac{du}{2}$

$= -\frac{1}{2} \cos x^2 + C'$

$\frac{d}{dx}\left(-\frac{1}{2}\cos x^2\right) =$

$-\frac{1}{2}\left(-\sin x^2\right) 2x$

$= x \sin x^2$

What about $\int_0^\pi x \sin x^2 \mathrm{d}x = \frac{1}{2} \int_0^{\pi^2} \sin u \, du = -\frac{1}{2} \cos u \Big|_0^{\pi^2}$

$u = x^2$

$x \, dx = \frac{1}{2} du$

$= -\frac{1}{2}\left(\cos \pi^2 - \cos 0\right)$

- Example 11, page 247
- $\int_0^{\pi/4} \sin^3 2x \cos 2x \mathrm{d}x =$

- Example 12, page 248
- $\int_0^1 \frac{x+1}{(x^2+2x+6)^2} \mathrm{d}x =$