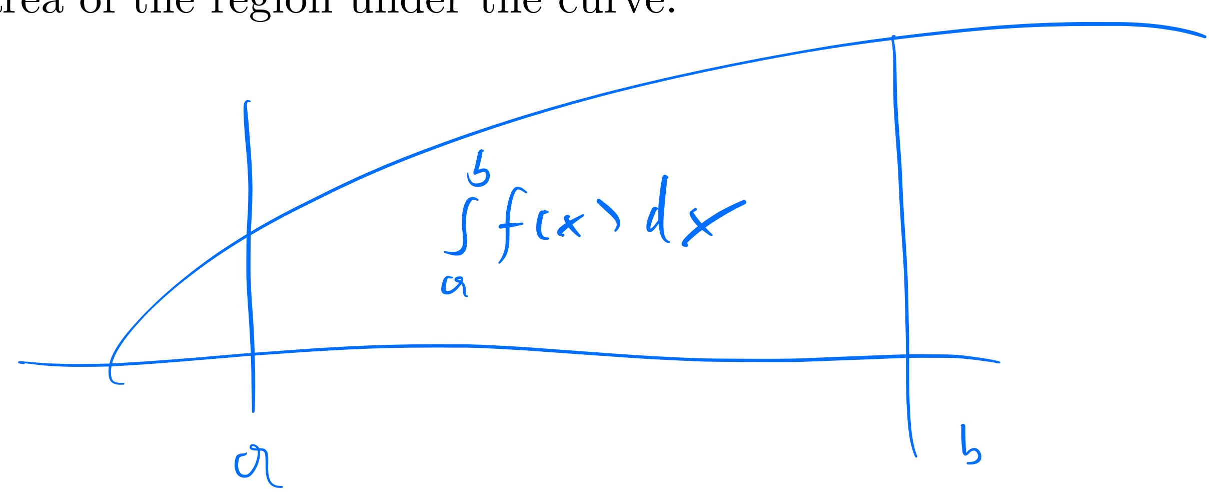
Math 1210-23

Notes of 3/22/24

Definite Integrals—Quick Review

- Recall that $\int_a^b f(x)\mathrm{d}x$ is the **definite integral of f with respect to x from a to b**.
- x is the **integration variable**.
- $f(x)$ is the **integrand**.
- a and b are the **lower and upper limits of integration**, respectively.
- We defined the definite integral as the limit of a **Riemann Sum**.
- Geometrically the definite integral gives the area of the region under the curve.



4.3-4.4 The Fundamental Theorem of Calculus

- We'll spend two days on sections 4.3 and 4.4 combined.
- Recall that we introduced derivatives and integrals by going back and forth between velocity and location.
- Naturally these two processes are inverses of each other.
- The Fundamental Theorem of Calculus (FToC) makes this precise.
- It comes in two flavors.
- Theorem A, page 235

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\mathrm{d}t = f(x)}$$

- Theorem A, page 243

$$\boxed{\int_a^b f(x)\mathrm{d}x = F(b) - F(a) \quad \text{where} \quad F'(x) = f(x).}$$

- The textbook calls these the first and second fundamental theorem of Calculus, but the two statements are equivalent and there is really only one FToC.
- We need to learn how to use these facts, and we need to see why they are true and why they are equivalent.
- Before thinking about why the statements are true and equivalent, let's do some examples.

FTOC

- Example

  $\int_0^2 x^2 \mathrm{d}x = F(2) - F(0) = \frac{8}{3}$

  $= \int_0^2 t^2 dt \qquad F'(x) = x^2 \qquad F(x) = \frac{x^3}{3} \qquad$ CBD

- Example:

  $\int_0^x t^2 \mathrm{d}t = \frac{x^3}{3}$

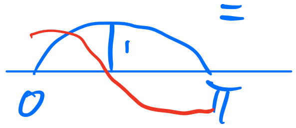
- Example:

  $\frac{\mathrm{d}}{\mathrm{d}x} \int_0^x t^2 \mathrm{d}t = \frac{d}{dx} \frac{x^3}{3} = x^2 \qquad \frac{d}{dx} \int_a^x f(t)\, dt = f(x)$

- Example:

  $\int_0^\pi \sin t \mathrm{d}t = F(\pi) - F(0) \qquad F(t) = -\cos t$

  $= 1 + 1 = 2$



- Example:

  $\int_1^3 3x + 4 \mathrm{d}x = \; = F(3) - F(1) = \frac{27}{2} + 12 - \left(\frac{3}{2} + 4\right) = \text{WTI}$

  $F(x) = \frac{3x^2}{2} + 4x$

- Example:

  $\int_3^5 4x^3 + 1 \mathrm{d}x = F(5) - F(3) = 5^4 + 5 - (3^4 + 3) = \text{WTI}$

  $F(x) = x^4 + x$

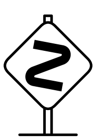
$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x t\mathrm{d}t = x$$

$$\int_0^x t\,dt = F(x) - F(0) = \frac{x^2}{2}$$

$$F(t) = \frac{t^2}{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \sin t\mathrm{d}t = \sin x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x t^t\mathrm{d}t = x^x$$



$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2} \sin t\mathrm{d}t = 2x\sin x^2$$

$$= \frac{d}{dx}\left(F(x^2) - F(0)\right) = \frac{d}{dx}\left(-\cos x^2\right) = 2x\sin x^2$$

$$F'(t) = \sin t$$

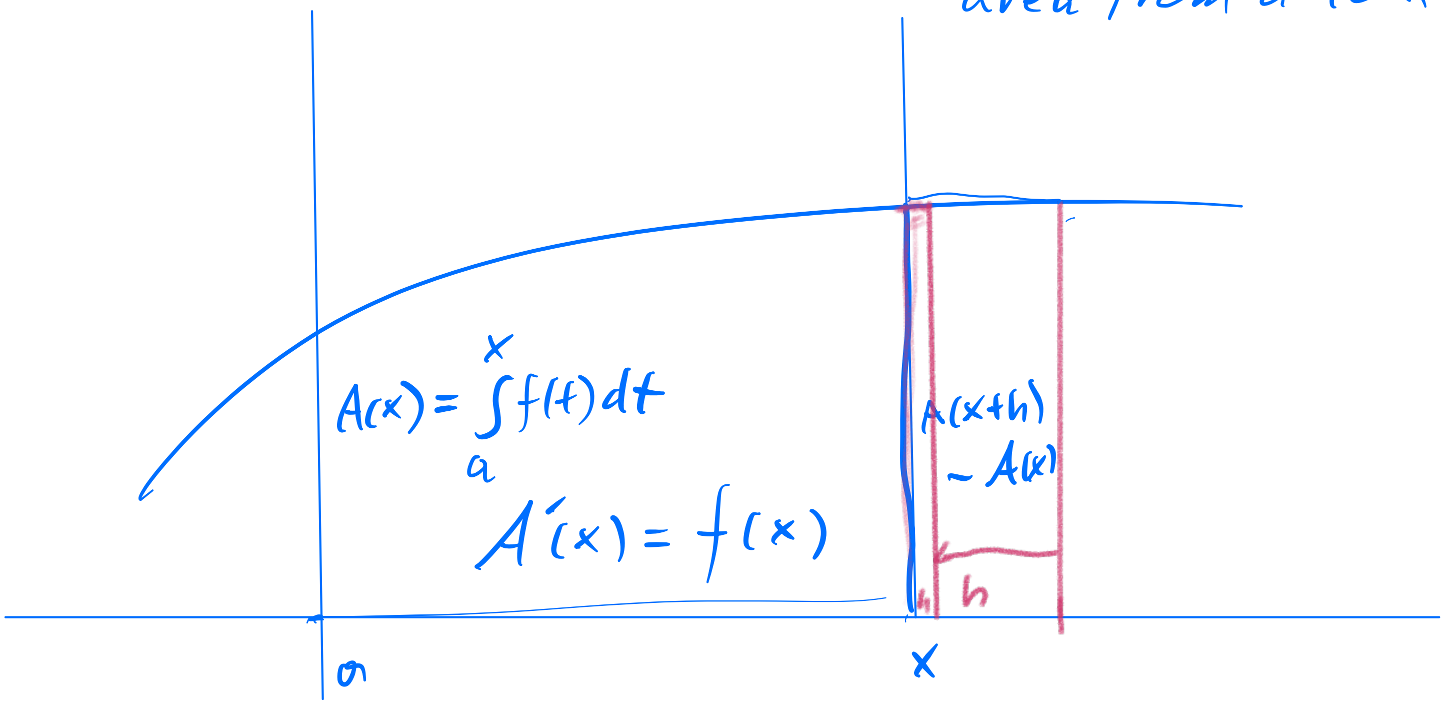
$$F(t) = -\cos t$$

Why is the FToC true?

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$$A(x) = \int_a^x f(t)\,dt$$

area from a to x



$$A'(x) = \lim_{h\to 0}\frac{A(x+h) - A(x)}{h} = f(x)$$

- The second version follows easily from the first.

$$\int_a^x f(t)\,dt = A(x) = F(x) - F(a)$$

$$A'(x) = f(x)$$

$$\frac{d}{dx}\Big(F(x) - F(a)\Big) = F'(x) = f(x)$$

$$C + F(x) = A(x) \qquad C = ?$$

$$F(a) + C = A(a) = 0$$

$$C = -F(a)$$

$$A(x) = F(x) + C$$

$$= F(x) - F(a)$$

$$A(b) = F(b) - F(a)$$

- The first version follows easily from the second.

$$\int_a^x f(t)\,dt = A(x) = F(x) - F(a)$$

$$\frac{d}{dx}\int_a^x f(t)\,dt = \frac{d}{dt} A(x) = \frac{d}{dx}\Big(F(x) - F(a)\Big)$$

$$= \frac{d}{dx} F(x)$$

$$= f(x)$$

- The two versions of the FToC are two sides of the same coin.

  There is only one FToC.

  Integration and Differentiation are opposite processes.

- You want to understand and remember those facts!

- Another Example.
- Suppose

$$G(x)=\int_0^{x^2} 2t+3t^2\mathrm{d}t.$$

- Compute $G'(x)$ in two different ways.

$$G(x) = F(x^2) - F(0) = (x^2)^2 + (x^2)^3 - 0$$
$$= x^4 + x^6 - 0$$

$$F'(t) = 2t + 3t^2$$
$$F(t) = t^2 + t^3$$

$$G(x) = x^4 + x^6$$
$$G'(x) = 4x^3 + 6x^5$$

$$G'(x) = \frac{d}{dx} F(x^2) - F(0)$$
$$= F'(x^2) \cdot 2x$$
$$= (2x^2 + 3x^4) 2x = 4x^3 + 6x^5$$

- Compare the previous example with

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x t^t \mathrm{d}t = x^x$$

- In this case we cannot first compute the definite integral.
- So what about

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2} t^t \mathrm{d}t = (x^2)^{(x^2)} \cdot 2x$$

$$\frac{d}{dx}\int_0^{\sin x} t^t dt = (\sin x)^{\sin x} \cdot \cos x$$

- In general we have

$$\boxed{\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_{L(x)}^{U(x)} f(t)\mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}x}\Big(F\big(U(x)\big) - F\big(L(x)\big)\Big)\\ &= f\big(U(x)\big)U'(x) - f\big(L(x)\big)L'(x)\end{aligned}}$$

- Don't try to memorize this formula. It comes straight from the FToC and the Chain Rule and is easy to derive when needed.

- More examples:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2} e^{t^2}\mathrm{d}t = e^{(x^2)^2}2x = 2xe^{x^4}$$

$$\int_0^{\pi/2} \sin t \cos t\mathrm{d}t = F\left(\frac{\pi}{2}\right) - F(0) = \frac{1}{2}$$

$$F(t) = \frac{1}{2}\sin^2 t$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\sin x} \cos t\mathrm{d}t = \cos(\sin x)\cdot\cos x$$

Notation

- Here are some frequent notations for the definite integral.
- Suppose F is an antiderivative of f, i.e.,

$$F' = f.$$

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$$

$$= [F(x)]_a^b$$

$$= [F(x)]_{x=a}^{x=b}$$

$$= F(x)\Big|_a^b$$

$$= F(x)\Big|_{x=a}^{x=b}$$

$$\int_1^2 x\,dx = \left[\frac{x^2}{2}\right]_1^2 = 2 - \frac{1}{2} = \frac{3}{2}$$