Math 1210-23

Notes of 3/22/24

Definite Integrals—Quick Review

- Recall that $\int_a^b f(x)\mathrm{d}x$ is the **definite integral of f with respect to x from a to** b.
- x is the **integration variable**.
- $f(x)$ is the **integrand**.
- a and b are the **lower and upper limits of integration**, respectively.
- We defined the definite integral as the limit of a **Riemann Sum**.
- Geometrically the definite integral gives the area of the region under the curve.

4.3-4.4 The Fundamental Theorem of Calculus

- We'll spend two days on sections 4.3 and 4.4 combined.
- Recall that we introduced derivatives and integrals by going back and forth between velocity and location.
- Naturally these two processes are inverses of each other.
- The Fundamental Theorem of Calculus (FToC) makes this precise.
- It comes in two flavors.
- Theorem A, page 235

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\mathrm{d}t = f(x)}$$

- Theorem A, page 243

$$\boxed{\int_a^b f(x)\mathrm{d}x = F(b) - F(a) \quad \text{where} \quad F'(x) = f(x).}$$

- The textbook calls these the first and second fundamental theorem of Calculus, but the two statements are equivalent and there is really only one FToC.
- We need to learn how to use these facts, and we need to see why they are true and why they are equivalent.
- Before thinking about why the statements are true and equivalent, let's do some examples.

- Example

  $\int_0^2 x^2 \mathrm{d}x =$

- Example:

  $\int_0^x t^2 \mathrm{d}t =$

- Example:

  $\frac{\mathrm{d}}{\mathrm{d}x} \int_0^x t^2 \mathrm{d}t =$

- Example:

  $\int_0^\pi \sin t \mathrm{d}t =$

- Example:

  $\int_1^3 3x + 4 \mathrm{d}x =$

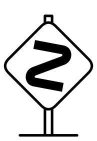
- Example:

  $\int_3^5 4x^3 + 1 \mathrm{d}x =$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x t\,\mathrm{d}t =$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \sin t\,\mathrm{d}t =$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x t^t\,\mathrm{d}t =$$



$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2} \sin t\,\mathrm{d}t =$$

Why is the FToC true?

- The second version follows easily from the first.

- The first version follows easily from the second.

- The two versions of the FToC are two sides of the same coin.

  There is only one FToC.

  Integration and Differentiation are opposite processes.

- You want to understand and remember those facts!

- Another Example.
- Suppose

$$G(x) = \int_0^{x^2} 2t + 3t^2 \mathrm{d}t.$$

- Compute $G'(x)$ in two different ways.

- Compare the previous example with

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x t^t \mathrm{d}t = x^x$$

- In this case we cannot first compute the definite integral.
- So what about

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2} t^t \mathrm{d}t = x^x$$

- In general we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{L(x)}^{U(x)} f(t)\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}x}\Big(F\big(U(x)\big) - F\big(L(x)\big)\Big)$$
$$= f\big(U(x)\big)U'(x) - f\big(L(x)\big)L'(x)$$

- Don't try to memorize this formula. It comes straight from the FToC and the Chain Rule and is easy to derive when needed.
- More examples:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2} e^{t^2}\mathrm{d}t =$$

$$\int_0^{\pi/2} \sin t \cos t\mathrm{d}t =$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\sin x} \cos t\mathrm{d}t =$$

Notation

- Here are some frequent notations for the definite integral.
- Suppose F is an antiderivative of f, i.e.,

$$F' = f.$$

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$$

$$= [F(x)]_a^b$$

$$= [F(x)]_{x=a}^{x=b}$$

$$= F(x)\Big|_a^b$$

$$= F(x)\Big|_{x=a}^{x=b}$$