Math 1210-4

Notes of 11/3/17

• Exam 3 next Wednesday, November 8, 2017

• Seven questions, on Chapter 3, Optimization, Mean Value Theorem, Shape of a graph versus derivatives, Graphing, word problems.

• Review next week on Monday and Tuesday.

• Review notes are now online.
4.1 Introduction to Area

• Recall the “indefinite integral” or “antiderivative”:

\[ \int f(x)\,dx = F(x) + C \quad \text{where} \quad F' = f. \]

• There is also a closely related **definite integral**.

• It gives the area underneath a curve.

• Recall that at the beginning of this semester we computed velocity by interpreting it as the area underneath a curve.
• We’ll now revisit (for the remainder of this semester) that concept.

• Basic idea: Approximate area by the area of a staircase, take the limit as the number of stairs goes to zero.

Figure 1. Riemann Sum.

Sum Notation

- Quick Review of **Sum** or **Sigma notation**

\[ \sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n = \sum_{j=1}^{n} a_j \]

introduced in College Algebra

- \( i \) or \( j \), the **summation index** has no meaning outsider the sum.

- \( \Sigma \) is the capital Greek letter Sigma.

- Here are some useful special formulas (textbook, p. 218):

\[ \sum_{i=1}^{n} 1 = n \]

\[ \sum_{i=1}^{n} i = \frac{n(n + 1)}{2} \]

\[ \sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6} \]

\[ \sum_{i=1}^{n} i^3 = \left( \frac{n(n + 1)}{2} \right)^2 \]
• Thus we approximate the area underneath a graph by the area of a staircase. Then we consider what happens as the number of stairs goes to infinity and the width of the stairs goes to zero.

$$\Delta x = \frac{b - a}{n}, \quad x_i = a + i\Delta x$$ \hspace{1cm} (1)

• the area of the box from $x_{i-1}$ to $x_i$ is $f(x_i)\Delta x$ and the total area is

$$A = \lim_{\Delta x \to 0} \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x.$$ \hspace{1cm} (2)
• The expression \( \sum_{i=1}^{n} f(x_i) \Delta x \) is an example of a Riemann Sum.

• Bernhard Riemann, 1826–1866.

• To get the standard definition of the Riemann Integral we need to get a little more general in the following ways:
  – allow for the subintervals to have different lengths
  – allow for evaluation of \( f \) at points other than the left endpoint of those subintervals
  – take the limit as the maximum size of those intervals goes to zero.
- Let's describe the idea graphically:

![Graph of Riemann Integral](image)

**Figure 3.** Riemann Integral.

- This gives rise to the celebrated definition on page 226 of our textbook.

\[ A = \int_a^b f(x)\,dx = \lim_{{\|P\| \rightarrow 0}} \sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i \quad (3) \]
• The expression \( \int_a^b f(x) \, dx \) is called the \textbf{definite integral} of \( f \) with respect to \( x \) from \( a \) to \( b \).

• If \( f \) is positive this is the area underneath the curve.

• \( x \) is the \textbf{integration variable}

• \( a \) and \( b \) are the \textbf{lower} and \textbf{upper limits of integration}.

• \( P \) is a \textbf{partition} of the interval \([a, b] \)

\[
P : \quad a = x_0 < x_1 < \ldots < x_n = b \quad (4)
\]

\[
\Delta x_i = x_i - x_{i-1}, \quad i = 1, 2, \ldots, n \quad (5)
\]

\[
\|P\| = \max_{i=1,\ldots,n} \Delta x_i \quad (6)
\]

\( \|P\| \) is called the \textbf{norm} of \( P \).

\( \bar{x}_i \) is any point in the interval \([x_{i-1}, x_i] \).

• Crucially, \textbf{the limit must be independent of how the partitions and the \( \bar{x}_i \) are chosen}.

• For a given function \( f \), the limit may not exist. If it does exist the function is \textbf{integrable}, otherwise it’s \textbf{non-integrable}. 
• What functions are integrable is a very subtle question, and a big issue in pure mathematics. For our purposes, which are focused on applications, it’s enough to know that any function that is continuous, or piecewise continuous, is also integrable.

• In our standard procedure
  vague notion $\rightarrow$ definition $\rightarrow$ properties
the above definition is the middle stage.

• Next week we will see that
  $$\int_{a}^{b} f(x) dx = F(b) - F(a) \quad \text{where} \quad F' = f$$

• So why, you ask, should we bother with the definition on the previous page?

• The value of that concept is the ability to go the other way: approximate something by a sum, recognize the limit of that sum as an integral, and then compute the integral.

• We will use that approach several times when we talk about applications for the rest of the semester.

• For that opposite approach we do not need to bother with intervals of differing lengths, and general evaluation points.

• But we do want to be able to interpret Riemann Sums as definite integrals.
Recall
\[ \Delta x = \frac{b - a}{n}, \quad x_i = a + i\Delta x \] (9)

\[ \int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^n f(x_{i-1})\Delta x. \] (10)

You want to be able to read this equation from right to left.

Examples

Example 1:
\[ \Delta x = \frac{3}{n}, \quad x_i = i\Delta x \] (11)

\[ \lim_{n \to \infty} \sum_{i=1}^n x_{i-1}^2 \Delta x = \int_a^b f(x)\,dx \] (12)

where
\[ a = 0 \]
\[ b = 3 \] (13)
\[ f(x) = x^2 \]
• Example 2:

\[ \Delta x = \frac{3}{n}, \quad x_i = 2 + i\Delta x \quad (14) \]

\[ \lim_{n \to \infty} \sum_{i=1}^{n} x_{i-1}^2 \Delta x = \int_{a}^{b} f(x) \, dx \quad (15) \]

where

\[ a = 2 \]

\[ b = 5 \quad (16) \]

\[ f(x) = x^2 \]
• Example 3:

\[ \Delta x = \frac{3}{n}, \quad x_i = -1 + i\Delta x \quad (17) \]

\[
\lim_{n \to \infty} \sum_{i=1}^{n} g^2(x_{i-1}) \Delta x = \int_{a}^{b} f(x) \, dx \quad (18) = \int_{-1}^{2} g^2(x) \, dx
\]

where

\[
g^2(x_{i-1}) = f(x_{i-1})
\]

\[ a = -1 \]

\[ b = 2 \quad (19) \]

\[ f(x) = g^2(x) \]
Example 4:

\[
\Delta x = \frac{2}{n}, \quad x_i = -1 + i\Delta x \quad (20)
\]

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 - x_i^2} \Delta x = \int_{a}^{b} f(x) \, dx \quad (21)
\]

where

\[
a = x_0 = -1
\]

\[
b = x_n = -1 + n \frac{2}{n} = 1
\]

\[
f(x) = \sqrt{1-x^2} \quad (22)
\]

\[
\int_{a}^{b} f(x) \, dx = \int_{-1}^{1} \sqrt{1-x^2} \, dx = \frac{\pi}{2}
\]

\[
\int_{a}^{b} f(x) \, dx = \int_{-1}^{1} \sqrt{1-x^2} \, dx = \frac{\pi}{2}
\]

\[
f(x_i) = \sqrt{1-x_i^2}
\]

\[
f(x) = \sqrt{1-x^2}
\]

\[
y = \sqrt{1-x^2} \Rightarrow y^2 = 1-x^2 \Rightarrow x^2 + y^2 = 1
\]
Properties

- Our definition can be generalized by dropping the requirement that \( a < b \). The norm of the partition is then the maximum length of a subinterval. Correspondingly, \( \Delta x_i \) may be zero, positive, or negative. As an exercise, think about the details.

- With this observation, the following properties of the definite integral follow straight from the

- Definition:

  \[
  \int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i \Delta x).
  \]  

- Zero-Length interval:

  \[
  \int_a^a f(x)\,dx = 0
  \]  

- Just like in a sum the choice of the summation index does not matter, in an integral the choice of the integration variable does not matter. Of course, you would not want to use the same variable as you use for the upper or lower limit of integration.

  \[
  \int_a^b f(x)\,dx = \int_a^b f(z)\,dz = \int_a^b f(t)\,dt
  \]
• just like a sum can be split:

\[
\sum_{i=1}^{n} a_i = \sum_{i=1}^{k} a_i + \sum_{i=k+1}^{n} a_i \quad (26)
\]

a definite integral can be split:

\[
\int_{a}^{b} f(x)dx = \int_{a}^{c} f(x)dx + \int_{c}^{b} f(x)dx \quad (27)
\]

• Integration is linear:

\[
\int_{a}^{b} \alpha f(t) + \beta g(t)dt = \alpha \int_{a}^{b} f(t)dt + \beta \int_{a}^{b} g(t)dt \quad (28)
\]

• Switching the limits of integration changes the sign of the definite integral

\[
\int_{a}^{b} f(t)dt = - \int_{b}^{a} f(t)dt \quad (29)
\]